Factorisation Homology and Skein Categories of Surfaces

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Juliet Cooke*)

This thesis is dedicated to my first supervisor the late Prof. Andrew Ranicki.

Abstract

In this thesis we show how skein algebras and skein categories can be computed by the mechanism of factorisation homology. We recover Kauffman bracket skein algebras of the four–punctured sphere and punctured torus from the presentable factorisation homology of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. Generalising this result, we then show that any skein category is a k–linear factorisation homology.

In the first part of this thesis, we study in detail the presentable factorisation homology of the four–punctured sphere and punctured torus with coefficients in the integrable representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. These factorisation homologies are categories of $\mathcal{U}_q(()\mathfrak{sl}_2)$–equivariant modules for algebras determined by the surface, and their $\mathcal{U}_q(\mathfrak{sl}_2)$–invariant subalgebra gives a quantisation of the SL_2–character variety of the surface. We obtain presentations and Poincaré–Birkhoff–Witt bases for the algebra of invariants for both our example surfaces. As an application, we explicitly identify these algebras of invariants with two other quantisations of the SL_2–character variety for these surfaces: Teschner and Vartanov's quantisation of the moduli space of flat connections and the Kauffman bracket skein algebra.

In the second part of this thesis, we pursue the relation between factorisation homology and skein theory further. We prove that skein categories satisfy excision and that they are k–linear factorisation homologies with coefficients given by the colouring of the skein category. As a corollary we show the free cocompletion of the skein category of the ribbon category of finite–dimensional representations of the quantum group $\mathcal{U}_q(\mathfrak{g})$ is the presentable factorisation homology with coefficients in the integrable representations of the quantum group $\mathcal{U}_q(\mathfrak{g})$. Hence, the free cocompletion of the Kauffman bracket skein category is the factorisation homology which we considered in the first part of the thesis.

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Chapter 1

Introduction

1.1 Factorisation Homology

Factorisation homology is a framework for constructing manifold–invariants by associating to a disc a system of local coordinates in an ∞-category and 'integrating' this object over the manifold. This association is achieved by the choice of an E_n–algebra. An E_n–algebra is an algebra over the little disc operad E_n in a symmetric monoidal ∞–category $\mathscr{C}^{\otimes}$, or equivalently it is a symmetric monoidal functor

$$F : \mathbf{Disc}_n^{\sqcup} \to \mathscr{C}^{\otimes} : F(D^n) = A \in \mathscr{C}^{\otimes}$$

from the ∞–category $\mathbf{Disc}_n$ of discs, embeddings and isotopies to $\mathscr{C}^{\otimes}$. The factorisation homology of the n–manifold M with coefficients in A is then an object $\int_M A \in \mathscr{C}^{\otimes}$ which is invariant up to homeomorphism of M.

Factorisation homology arose from the chiral homology of Beilinson and Drinfeld [BD04]. Chiral homology was adapted from a conformal to a topological setting by Lurie [Lur17]. This topological chiral homology was developed further by Ayala, Francis and Tanaka who rechristened it factorisation homology [AF15, AFT17].

Ayala and Francis showed that factorisation homologies satisfy a generalisation of the Eilenberg-Steenrod axioms for singular homology [AF15], so may be interpreted as a generalisation of homology which is tailor–made for topological manifolds rather than general topological spaces. In particular, factorisation homologies satisfy excision. Certain factorisation homologies are known to recover other homology theories, for example if A is an abelian group then $\int_M A$ is simply given by the singular homology $H_*(M, A)$, and if A is an associative algebra then $\int_{S^1}(A)$ is the Hochschild homology $HH_\bullet(A)$; see [AF19] for elaboration and further examples.

1.2 Topological Quantum Field Theory

A major motivation for the development of factorisation homology comes from topological quantum field theory. Topological quantum field theory was inspired by Witten's formulation of supersymmetric quantum field theories in terms of the differential geometry of certain infinite–dimensional manifolds [Wit82]. Topological quantum field theories are toy–model quantum field theories: non-relativistic topologically invariant quantum field theories where the manifolds are assumed to be finite–dimensional. Their mathematical formulation was developed by Atiyah [Ati88] who modelled the definition on Segal's formulation for conformal field theory [Seg88]. A n-dimensional topological quantum field theory (TQFT) is a symmetric monoidal functor $Z : \mathbf{Bord}_n^{\sqcup} \to \mathscr{C}^{\otimes}$ from the bordism category $\mathbf{Bord}_n^{\sqcup}$, whose objects are closed $(n-1)$–dimensional manifolds and whose morphisms are n-dimensional cobordisms, to some symmetric monodial category $\mathscr{C}^{\otimes}$ which was classically chosen to be the category of vector spaces $\mathbf{Vect}_k$. Despite TQFTs being physically toy–models, they are of significant interest in low dimensional topology as the assignment $M \mapsto Z(M)$ defines a topological invariant of the closed manifold M. These invariants are sometimes classical invariants of low dimensional topology, for example, the 3-dimensional Chern–Simons TQFT recovers the Jones polynomial and the 4–dimensional supersymmetric gauge theory TQFT recovers Donaldson invariants.

One can extend the definition of a n–dimensional TQFT by replacing $\mathscr{C}^{\otimes}$ with a suitable symmetric monoidal n–category and defining an n–categorical version of $\mathbf{Bord}_n^{\sqcup}$ with the n–morphisms being n–dimensional cobordisms between $(n-1)$–dimensional manifolds, the $(n-1)$-morphisms being $(n-1)$–dimensional cobordisms between $(n-2)$–dimensional manifolds, and so on until one reaches 0–dimensional manifolds, i.e. points, which are the objects of $\mathbf{Bord}_n^{\sqcup}$. A fully extended 2–dimensional TQFT differs from an ordinary 2–dimensional TQFT by allowing surfaces with corners. Baez and Dolan [BD04] conjectured that these fully extended TQFTs are fully determined by their value at a point and that every fully dualisable object gives rise to a fully extended TQFT. This is called the Cobordism Hypothesis and a sketch proof of it was provided by Lurie [Lur09]. By the Cobordism Hypothesis, to define a fully extended TQFT it is enough to define a fully dualisable object; however, using this formulation it it far from clear how this TQFT acts on manifolds. Scheimbauer shows that one can use n–dimensional factorisation homology to construct a fully extended TQFT [Sch14].

1.3 Quantum Character Varieties

We now turn from considering general factorisation homologies of manifolds to the factorisation homologies of surfaces with coefficients in the representations of quantum groups.

Fix a connected reductive Lie group G such that its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is semisimple. Drinfeld defined a quantisation $\mathcal{U}_q(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$ which is called the quantum group of $\mathfrak{g}$ [Dri87]. Throughout this thesis we shall assume that $q \in \mathbb{C}$ is generic i.e. not a root of unity. We define $\mathbf{Rep}_q(G)$ to be the category of integrable representations of $\mathcal{U}_q(\mathfrak{g})$. Ben-Zvi, Brochier, and Jordan show that the factorisation homology $\int_{\Sigma_0} \mathbf{Rep}_q(G)$ of the punctured surface Σ_0 is equivalent to the category A_{Σ_0}âĂŤ-*mod* of internal modules for some algebra object A_{Σ_0} which is determined combinatorially from the gluing pattern of Σ_0 [BZBJ18a]. They also relate the factorisation homology $\int_{\Sigma} \mathbf{Rep}_q(G)$ of a non-punctured surface Σ to the punctured case [BZBJ18b].

The representation variety $\mathfrak{R}_G(\Sigma)$ of the surface Σ consists of all the homomorphisms from the fundamental group $\pi_1(\Sigma)$ to the connected reductive Lie group G. There are two widely studied invariants of Σ based on the representation variety: the character stack $\underline{\mathrm{Ch}}_G(\Sigma) = \mathfrak{R}_G(\Sigma)/G$ which is the quotient of the representation variety by G which acts on it by conjugation, and the character variety $\mathrm{Ch}_G(\Sigma) = \mathfrak{R}_G//G$ which instead takes the affine categorical quotient.

Ben-Zvi, Brochier and Jordan show that a quantisation of the character stack $\underline{\mathrm{Ch}}_G(\Sigma)$ is given by the algebra object A_Σ for punctured surfaces and a Hamiltonian reduction of this algebra object for closed surfaces [BZBJ18a, BZBJ18b]. However, in this thesis we shall instead concern ourselves with quantisations of the character variety $\mathrm{Ch}_G(\Sigma)$.

The character variety $\mathrm{Ch}_G(\Sigma)$ has a canonical Poisson structure which was defined by Atiyah–Bott and Goldman [AB83, Gol84], so by a quantisation of the character variety we mean a deformation with respect to this Poisson bracket. Ben-Zvi, Brochier, and Jordan [BZBJ18a] show that $\mathscr{A}_{\Sigma_0} = (\underline{\mathrm{End}}(A_{\Sigma_0}))^{\mathcal{U}_q(()\mathfrak{g})}$, the algebra of invariants of A_{Σ_0} under the action of $\mathcal{U}_q(()\mathfrak{g})$, is a quantisation of the character variety $\mathrm{Ch}_G(\Sigma_0)$ of the punctured surface Σ_0 [BZBJ18a].

The main result of Chapter 3 is finding this algebra of invariants for the four–punctured sphere $\Sigma_{0,4}$ and punctured torus $\Sigma_{1,1}$ with respect to $\mathcal{U}_q(\mathfrak{sl}_2)$:

Theorem 1.3.1. *Let* $A := \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{smallmatrix}\right)$ $B := \left(\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{smallmatrix}\right)$ *and* $C := \left(\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{smallmatrix}\right)$ *be the matrices formed out of the 12 generators of* $\int_{\Sigma_{0,4}} \mathbf{Rep}_q(\mathrm{SL}_2)$. *The algebra of invariants* $\mathscr{A}_{\Sigma_{0,4}}$ *of the four-punctured sphere with respect to* $\int_\Sigma \mathbf{Rep}_q(G)$ *has a presentation with generators the quantum traces* $E := \mathrm{Tr}_q(AB)$, $F := \mathrm{Tr}_q(AC)$, $G := \mathrm{Tr}_q(BC)$, $s := \mathrm{Tr}_q(A)$, $t := \mathrm{Tr}_q(B)$, $u := \mathrm{Tr}_q(C)$ $v := \mathrm{Tr}_q(ABC)^\dagger$, *and relations*

$$\begin{aligned} FE &= q^2EF + \quad (q^2-q^{-2})G + (1-q^2)(sv+tu), \\ GE &= q^{-2}EG + q^{-2}(q^2-q^{-2})F - (1-q^2)(su+q^{-2}tv), \\ GF &= q^2FG + \quad (q^2-q^{-2})E + (1-q^2)(st+uv), \\ EFG &= \begin{cases} -E^2 - q^{-4}F^2 - G^2 - q^{-4}(s^2+t^2+u^2+v^2) \\ +(st+uv)E + q^{-2}(su+tv)F + (sv+tu)G \\ -stuv + q^{-6}(q^2+1)^2 \end{cases} \end{aligned}$$

and s, t, u, v *are central. Furthermore, the monomials*

$$\left\{ E^mF^nG^ls^at^bu^cv^d \;\middle|\; m,n,l,a,b,c,d \in \mathbb{N}_0; mnl = 0 \right\}$$

are a Poincaré-Birkhoff-Witt (PBW) basis for the algebra.

Theorem 1.3.2. *Let* $A := \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{smallmatrix}\right)$ *and* $B := \left(\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{smallmatrix}\right)$ *be the matrices formed out of the 8 generators of* $\int_{\Sigma_{1,1}} \mathbf{Rep}_q(\mathrm{SL}_2)$. *The algebra of invariants* $\mathscr{A}_{\Sigma_{1,1}}$ *of the punctured torus with respect to* $\int_\Sigma \mathbf{Rep}_q(G)$ *has a presentation given by generators* $X := \mathrm{Tr}_q(A), Y := \mathrm{Tr}_q(B), Z := \mathrm{Tr}_q(AB)$ *and relations:*

$$\begin{aligned} YX - q^{-1}XY &= (q-q^{-1})Z; \\ XZ - q^{-1}ZX &= -q^{-3}(q-q^{-1})Y; \\ ZY - q^{-1}YZ &= -q^{-3}(q-q^{-1})X. \end{aligned}$$

It has a central element

$$L := q^5XZY + q^3Y^2 - q^4Z^2 + q^3X^2 - (q-q^{-1}),$$

†They correspond to loops around two punctures as depicted for in Figure 3.5 for $A = x_1$, $B = x_2$ and $C = x_3$.

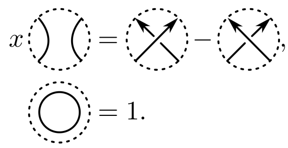
and a PBW basis

$$\left\{ X^\alpha Y^\beta Z^\gamma \mid \alpha, \beta, \gamma \in \mathbb{N}_0 \right\}.$$

Besides this algebra of invariants, there are other known quantisations of $\mathrm{Ch}_G(\Sigma)$ most especially when $G = \mathrm{SL}_2$. We can use the presentations from Theorems 1.3.1 and 1.3.2 to compare different quantisations for these two example surfaces. One quantisation of $\mathrm{Ch}_{\mathrm{SL}_2}(\Sigma)$ is given by Teschner and Vartanov's quantisation of the moduli space of flat connections $\mathcal{A}_b(\Sigma)$ which uses the four–punctured sphere and punctured torus as base cases [VT13]. In Chapter 3 we construct isomorphisms between the algebra of invariants $\mathscr{A}_\Sigma$ and $\mathcal{A}_b(\Sigma)$ for both surfaces. Another particularly interesting quantisation of the character variety $\mathrm{Ch}_{\mathrm{SL}_2}(\Sigma)$ is given by the Kauffman bracket skein algebra.

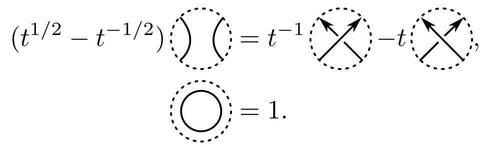
1.4 Skein Algebras

Skein algebras and modules are a generalisation of knot polynomials. A knot polynomial is a knot invariant which to each link L assigns an ordinary or Laurent polynomial. The first knot polynomial was the Alexander polynomial $\Delta_L(x)$ which was defined in 1928 [Ale28], and for almost 50 years it remained the only knot polynomial. In 1969 Conway [Con70] showed that Alexander polynomial $\Delta_L(x)$ was characterised by the *skein relations*



In a couple of talks at the end of the 70s he proposed considering the free $\mathbb{Z}[z]$–module over oriented links in a oriented 3–manifold, and the submodule generated by quotienting by the skein relations: he called this submodule the *linear skein module*; however, he published nothing. This idea was then developed by Giller, Kauffman, Lickorish, Millett, Przytycki and Turaev during the 80s [Gil82, Kau83, Kau87, LM87, Prz91, Tur97]. A skein module may be viewed as a generalisation of the 1st homology group of a manifold where the cycles have been replaced with general links.

A major impetus for the study of skein relations was the discovery of the Jones polynomial [Jon97], and in particular Jones' realisation that the Jones polynomial is characterised by the skein relations

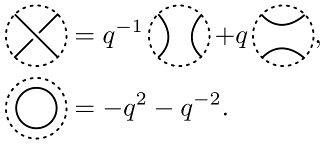


This quickly lead to the HOMPFLY polynomial which simultaneously generalised both the Alexander and Jones polynomial and is also defined in terms of skein relations [FYH$^+$85]. For a general survey see [Prz06].

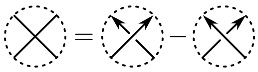
As has already been mentioned the Kauffman bracket skein algebra gives a quantisation of the character variety $\mathrm{Ch}_{\mathrm{SL}_2}(\Sigma)$ [Bul97, PS00]†. If we have a surface Σ, we can define its skein

†This is actually a result about Kauffman bracket skein modules and quantisations of the character variety $\mathrm{Ch}_{\mathrm{SL}_2}(M)$ of the 3–dimensional manifold M, but we shall only consider surfaces in this thesis. It has also been generalised to SL_N using the HOMPFLY skein modules by Sikora [Sik05].

algebra to be the skein module of $\Sigma \times [0,1]$; it has a natural algebra structure given by stacking. The Kauffman bracket skein algebra/module is based on the Kauffman bracket polynomial. The Kauffman bracket polynomial $\langle L \rangle$ of a framed link L is defined using the following skein relations



It is an invariant of framed links i.e. it is invariant under the 2nd and 3rd Reidemeister moves but not the 1st. The Kauffman bracket polynomial can be normalised to make it invariant under the 1st Reidemeister move and this recovers the Jones polynomial. It can also been extended using the relation



to give a Vassiliev invariant of singular framed knots.

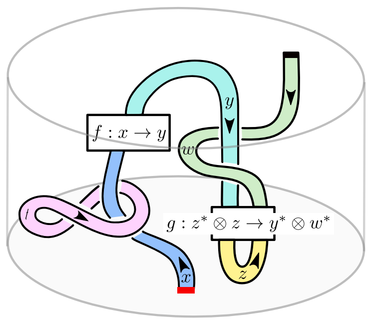
We show in Chapter 3 that

Proposition 1.4.1. *The algebra of invariants $\mathscr{A}_\Sigma$ with respect to $\mathcal{U}_q(\mathfrak{sl}_2)$ is isomorphic to the Kauffman bracket skein algebra* $\mathrm{Sk}(\Sigma)$ *when Σ is the four–punctured sphere or the punctured torus.*

In Chapter 3 we also construct the isomorphisms.

1.5 Skein Categories

After showing the relation of $\int_\Sigma^{\mathbf{Pr}} \mathbf{Rep}_q(\mathrm{SL}_2)$ to the Kauffman bracket skein algebra $\mathrm{Sk}(\Sigma)$ for the surfaces $\Sigma = \Sigma_{0,4}$ and $\Sigma_{1,1}$ via the algebra of invariants of the factorisation homology, we move on to the more general question: Is there any general relation between $\int_\Sigma^{\mathbf{Pr}} \mathbf{Rep}_q(G)$ and skein theory? In order to answer this question we introduce the skein category $\mathbf{Sk}_\mathscr{V}(\Sigma)$ for a fixed k–linear ribbon category $\mathscr{V}$ such as $\mathbf{Rep}_q^{\mathrm{fd}}(G)$ the category of finite–dimensional integrable representations of $\mathcal{U}_q(\mathfrak{sl}_2)$. The notion of skein category we use is that of Johnson-Freyd [Joh15] which was inspired by the ideas of Walker [Wal06, MW11] and Turaev's ribbon diagram category [Tur94, Tur97]. The ribbon diagram category $\mathbf{Ribbon}_\mathscr{V}$ is the category of $\mathscr{V}$–coloured ribbon tangles in $[0,1]^3$ and were originally developed in the context of the Reshetikhin–Turaev invariants for 3–manifolds. Turaev shows that there is a canonical surjective and full ribbon functor $eval : \mathbf{Ribbon}_\mathscr{V} \to \mathscr{V}$. The skein category $\mathbf{Sk}_\mathscr{V}(\Sigma)$ is the k–linear category whose



Example of a coloured ribbon diagram in $[0,1]^3$.

1. Objects are finite sets of framed points in Σ;

2. Morphisms are k–linear combinations of $\mathscr{V}$–coloured ribbon tangles in $\Sigma \times [0,1]$ up to the equivalence that $F \sim G$ if they are equal outside a cube and $eval(F|_{cube}) = (G|_{cube})$.

For a more precise definition see Section 4.2.1.

As we have already mentioned, one of the defining features of a factorisation homology $\int_\Sigma \mathscr{V}$ is that it satisfies excision: for any collar gluing $\Sigma = M \cup_A N$ there is an equivalence of categories

$$\int_{M \cup_A N} \mathscr{V} \simeq \int_M \mathscr{V} \otimes_{\int_A \mathscr{V}} \int_N \mathscr{V},$$

where the relative tensor product is defined as the colimit of the 2–sided bar construction in the ambient category $\mathscr{C}^\otimes$.

In Chapter 4 we show that if we take $\mathscr{C}^\otimes$ to be $\mathbf{Cat}_k^\times$, the $(2,1)$–category of k–linear categories, then this relative tensor product $\mathscr{M} \times_{\mathscr{A}} \mathscr{N}$ is $\mathscr{M} \times \mathscr{N}$ with adjoined isomorphisms $\iota : (m \lhd a, n) \to (m, a \rhd n)$ which relate the action of $\mathscr{A}$ on $\mathscr{M}$ to its action on $\mathscr{N}$ (see Section 4.1 for details). As the skein category $\mathbf{Sk}_{\mathscr{V}}(\Sigma)$ is a k–linear category this defines the relative tensor product of skein categories. We then prove that skein categories satisfy excision:

Theorem 1.5.1. *For any collar gluing $\Sigma = M \cup_A N$ there is an equivalence of categories*

$$\mathbf{Sk}_{\mathscr{V}}(M \cup_A N) \simeq \mathbf{Sk}_{\mathscr{V}}(M) \otimes_{\mathbf{Sk}_{\mathscr{V}}(A)} \mathbf{Sk}_{\mathscr{V}}(N).$$

Using this we conclude

Theorem 1.5.2. *The functor $\mathbf{Sk}_{\mathscr{V}}(\_) : \mathbf{Mfld}_{\mathrm{fr}}^{\sqcup} \to \mathbf{Cat}_k^\times$ is the k–linear factorisation homology $\int_{\_}^{\mathbf{Cat}_k} \mathscr{V}$ with respect to the E_2–algebra defined by $\mathscr{V}$.*

Corollary 1.5.3. *The free cocompletion of $\mathbf{Sk}_{\mathbf{Rep}_q^{\mathrm{fd}}(G)}(\Sigma)$ is the presentable factorisation homology $\int_\Sigma^{\mathbf{Pr}} \mathbf{Rep}_q(G)$.*

The excision of skein categories was conjectured by Johnson-Freyd [Joh15] again based on the ideas of Walker [Wal06, MW11] and the relation to presentable factorisation homology by taking the free cocompletion was conjectured in [BZBJ18a]. The is also a result of Yetter [Yet92] which proves a similar excision result for universal braid categories in **Set**, and the topological parts of the proof of excision are based on this proof.

Chapter 2

Background

2.1 The Categories $\mathbf{Cat}_k$ and $\mathbf{Pr}$

In this section we shall define two $(2,1)$–categories $\mathbf{Cat}_k$ and $\mathbf{Pr}$ which will be the ambient categories of the factorisation homologies considered in this thesis. The definitions in this section may be found in Borceux's 'Handbook of Categorical Algebra' [Bor94a, Bor94b] and follow the terminology of [BZBJ18a].

Definition 2.1.1. A $(2,1)$–category $\mathscr{C}$ is a 2–category for which all 2–morphisms have inverses.

Remark 2.1.2. There are two notions of 2–category: strict and weak. Throughout this thesis we shall assume all 2–categories are strict i.e. categories enriched over **Cat**. We shall refer to weak 2–categories by their original name of bicategories. However, ∞–categories may strict or weak.

2.1.1 The Category $\mathbf{Cat}_k$

Definition 2.1.3. Let k be a commutative ring with identity. The category $k\mathbf{Mod}$ is the category of left k–modules and module homomorphisms. If k is a field then $k\mathbf{Mod}$ is $\mathbf{Vect}_k$, the category of k–vector spaces and k–linear transformations.

Definition 2.1.4. A *k–linear category* is a category enriched over $k\mathbf{Mod}$, a *k–linear functor* is a $k\mathbf{Mod}$–enriched functor, and a *k–linear natural transformation* is a $k\mathbf{Mod}$–enriched natural transformation.

Definition 2.1.5. The *category of k–linear categories* $\mathbf{Cat}_k$ is the $(2,1)$–category whose

1. objects are small k–linear categories;
2. 1–morphisms are k–linear functors;
3. 2–morphisms are k–linear natural isomorphisms.

2.1.2 The Category $\mathbf{Pr}$

We shall begin by defining **Cocomp** which has **Pr** as a subcategory.

Definition 2.1.6. Given k–linear functors $F : \mathscr{D} \to \mathscr{C}$ and $G : \mathscr{D}^{op} \to k\mathbf{Mod}$, let $\mathrm{Colim}_G(F)$ denote the k–linear colimit of F weighted by G.

Definition 2.1.7. The k–linear category $\mathscr{C}$ is *cocomplete* if the colimit $\mathrm{Colim}_G(F)$ exists for all choices of F and G when $\mathscr{D}$ is small.

Definition 2.1.8. A functor $H : \mathscr{C} \to \mathscr{E}$ *preserves* the k–linear colimit of $F : \mathscr{D} \to \mathscr{C}$ weighted by $G : \mathscr{D}^{op} \to \mathscr{C}$ if

$$H\left(\mathrm{Colim}_G(F)\right) = \mathrm{Colim}_G(H(F)).$$

A functor is *cocontinuous* if it preserves all small limits.

Definition 2.1.9. We denote by **Cocomp** the $(2,1)$-category with:

1. objects: locally small cocomplete k–linear categories;
2. 1-morphisms: cocontinuous k–linear functors;
3. 2-morphisms: k–linear natural isomorphisms.

The subcategory $\mathbf{Pr} \subset \mathbf{Cocomp}$ consists of categories whose objects are 'nice' colimits of 'small' objects.

Definition 2.1.10. A category $\mathscr{C}$ is *filtered* if

1. $\mathscr{C}$ is non-empty;
2. For any two objects $c_1, c_2 \in \mathscr{C}$ there exists an object $c_3 \in \mathscr{C}$ with morphisms $c_1 \to c_3$ and $c_2 \to c_3$;
3. For any two morphisms $f, g : c_1 \rightrightarrows c_2$ there is a morphism $h : c_2 \to c_3$ such that $h \circ f = h \circ g$.

A *filtered colimit* $\mathrm{Colim}_G(F)$ is a colimit where $\mathscr{D}$ is a small filtered category.

Definition 2.1.11. An object $c \in \mathscr{C}$ of a k–linear category $\mathscr{C}$ is *finitely presentable* or *compact* if the corepresentable functor $\mathscr{C}(c, \_) : \mathscr{C} \to k\mathbf{Mod}$ preserves filtered colimits.

Definition 2.1.12. A category $\mathscr{C}$ is *locally finitely presentable* if it is a locally small, cocomplete and is generated under filtered colimits by a set of finitely presentable objects.

Remark 2.1.13. There is also a notion of a *locally presentable* category. A locally presentable category is a category which is locally small, cocomplete and is generated under κ-filtered colimits by a set of κ-compact objects for some regular cardinal κ. A locally finitely presentable category is a locally presentable category with $\kappa = \aleph_0$. By presentable we shall always mean locally finitely presentable unless stated otherwise.

Definition 2.1.14. A functor $F : \mathscr{C} \to \mathscr{D}$ is *compact* if it preserves compact objects i.e. if c is a compact object of $\mathscr{C}$ then $F(c)$ is a compact object of $\mathscr{D}$.

Definition 2.1.15. Let **Pr** denote the subcategory of **Cocomp** with:

1. objects: locally finitely presentable k–linear categories;
2. 1-morphisms: compact cocontinuous k–linear functors;
3. 2-morphisms: k–linear natural isomorphisms.

2.2 Monoidal and Ribbon Categories

Definition 2.2.1. [Bor94b] A *monoidal linear category* $\mathscr{C}$ is a k–linear category equipped with

1. a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C} : (a,b) \mapsto a \otimes b$ called the *monoidal product* or *tensor product*;
2. an object $1_{\mathscr{C}} \in \mathscr{C}$ called the *monoidal unit*;
3. a natural isomorphism $\alpha : (\_ \otimes \_) \otimes \_ \to \_ \otimes (\_ \otimes \_)$ with components $\alpha_{a,b,c} : (a \otimes b) \otimes c \to a \otimes (b \otimes c)$ called the *associator*;
4. natural isomorphisms

$$\lambda : 1_{\mathscr{C}} \otimes \_ \to \_ \text{ with components } \lambda_a : 1_{\mathscr{C}} \otimes a \to a,$$
$$\rho : \_ \otimes 1_{\mathscr{C}} \to \_ \text{ with components } \rho_a : a \otimes 1_{\mathscr{C}} \to a$$

called the *left and right unitor* respectively

which make the following diagrams commute for all $a, b, c, d \in \mathscr{C}$

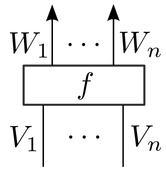
$$\begin{tikzcd}
 & ((a\otimes b)\otimes c)\otimes d \arrow[dl, "\alpha_{a,b,c}\otimes \mathrm{Id}_d"'] \arrow[dr, "\alpha_{a\otimes b,c,d}"] & \\
(a\otimes(b\otimes c))\otimes d \arrow[d, "\alpha_{a,b\otimes c,d}"'] & & (a\otimes b)\otimes(c\otimes d) \arrow[d, "\alpha_{a,b,c\otimes d}"] \\
a\otimes((b\otimes c)\otimes d) \arrow[rr, "\mathrm{Id}_a \otimes\alpha_{b,c,d}"] & & a\otimes(b\otimes(c\otimes d))
\end{tikzcd}
\qquad
\begin{tikzcd}
(a\otimes 1_{\mathscr{C}})\otimes b \arrow[r, "\alpha_{x,1_{\mathscr{C}},y}"] \arrow[d, "\rho_a\otimes\mathrm{Id}_b"'] & a\otimes(1_{\mathscr{C}}\otimes b) \arrow[dl, "\mathrm{Id}_a\otimes\lambda_b"] \\
x\otimes y &
\end{tikzcd}$$

The monoidal category may be denoted $\mathscr{C}$ or $\mathscr{C}^{\otimes}$. If the associator and unitors are trivial then the monoidal category is *strict.*

Remark 2.2.2. If $\mathscr{C}$ is a category enriched over the monoidal category $\mathscr{V}$, then we require that the monoidal structure is compatible with the enrichment, that is we require $\otimes$ to be a $\mathscr{V}$–enriched functor, and α and λ to be $\mathscr{V}$–enriched natural transformations. So if $\mathscr{C}$ is a k–linear category then we require $\otimes$, α, λ and ρ to be k–linear, and if $\mathscr{C}$ is a 2–category then we require $\otimes$ to be a (strict) 2–functor, and λ and ρ to be 2–natural transformations.

Remark 2.2.3. In this thesis our examples of ribbon categories will be non–strict; however, our applications of ribbon categories, for example to colour ribbons, will require strict monoidal categories. This is resolved by taking the monoidally equivalent strict ribbon category whenever a strict ribbon category is required, and this we shall do without further comment.

Remark 2.2.4. Monoidal categories[†] have a diagrammatic calculus in which the morphism $f : V_1 \otimes \cdots \otimes V_n \to W_1 \otimes W_2$ is depicted



[†]Technically the diagrammatic calculus is for strict monoidal categories.

The identity morphism is depicted without a coupon, the composition of morphisms is depicted by stacking and tensoring of morphisms is depicted by placing the diagrams side–by–side. For a survey on the diagrammatic calculi of monoidal categories see [Sel11].

We shall now define the structures required to turn the k–linear monoidal category $\mathscr{C}$ into a k–linear ribbon category.

Definition 2.2.5 [Kas95]**.** Let $\mathscr{C}$ be a k–linear monoidal category. The *flip functor* on the category $\mathscr{C}$ is the k–linear functor

$$\tau : \mathscr{C} \times \mathscr{C} \to \mathscr{C} \times \mathscr{C} : \tau(a,b) = (b,a) \text{ and } \tau(f,g) = (g,f)\ \forall a,b \in \mathscr{C} \text{ and } f,g \text{ morphisms in } \mathscr{C}.$$

A *braiding* on $\mathscr{C}$ is a k–linear natural isomorphism $B : \otimes \to \tau\otimes$ which is compatible with the monoidal structure:

$$\alpha_{b,c,a} \circ B_{a,b\otimes c} \circ \alpha_{a,b,c} = \mathrm{Id}_b \otimes B_{a,c} \circ \alpha_{b,a,c} \circ B_{a,b} \otimes \mathrm{Id}_c$$
$$\alpha^{-1}_{c,a,b} \circ B_{a\otimes b,c} \circ \alpha^{-1}_{a,b,c} = B_{a,c} \otimes \mathrm{Id}_b \circ \alpha^{-1}_{a,c,b} \circ \mathrm{Id}_a \otimes B_{b,c}$$

for all $a,b,c \in \mathscr{C}$. A monoidal category with a braided is called a *braided monoidal category.* A *symmetric monoidal category* $\mathscr{C}$ is a braiding monoidal category for which the braiding satisfies

$$B_{y,x} B_{x,y} = \mathrm{Id}_{x\otimes y}$$

for all $x,y \in \mathscr{C}$.

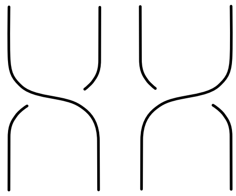


Figure 2.1: The diagrammatic calculus for monoidal categories can be adapted to give a diagrammatic calculus for braided monoidal categories for further details see [JS93, Sel11]. The braiding $B_{V,W}$ is depicted of the left and its inverse $B^{-1}_{V,W}$ is depicted on the right.

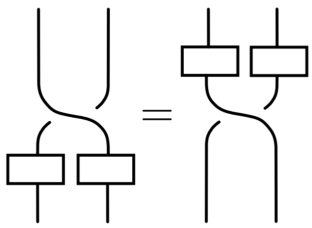


Figure 2.2: The naturality of the braiding means that coupons may pass through the braiding.

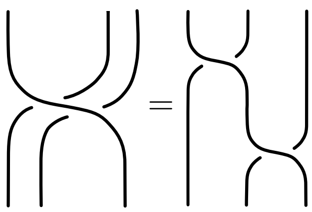


Figure 2.3: For a strict monoidal category, the first associativity condition on the braiding reduces to $B_{U,V\otimes W} = (Id_V \otimes B_{U,W}) \circ (B_{U,V} \otimes Id_W)$. This means that strands can always be crossed pairwise (the second associativity condition is just the mirror image).

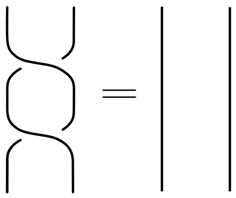


Figure 2.4: A monoidal category is symmetric if this identity holds.

Definition 2.2.6 [Tur94]**.** Let $\mathscr{C}$ be a k–linear monoidal category and $a \in \mathscr{C}$. If it exists, the dual of a is an object a^* such that there are two morphisms

$$\epsilon_a : 1_{\mathscr{C}} \to a \otimes a^* (\text{unit}) \text{ and } \eta_a : a^* \otimes a \to 1_{\mathscr{C}} \ (\text{counit})$$

which satisfy the following identities

$$(\mathrm{Id}_a \otimes \eta_a)(\eta_a \otimes \mathrm{Id}_a) = \mathrm{Id}_a,$$
$$(\eta_a \otimes \mathrm{Id}_{a^*})(\mathrm{Id}_{a^*} \otimes \epsilon_a) = \mathrm{Id}_{a^*} \ \ (\text{zigzag identities}).$$

A monoidal category has *duality* is every object has a dual.

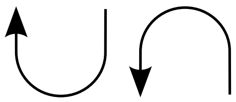


Figure 2.5: The dual of an object V is depicted either by labelling the strand with V^* or reversing the direction of the strand and labelling it with V. The unit and counit are depicted in this figure.

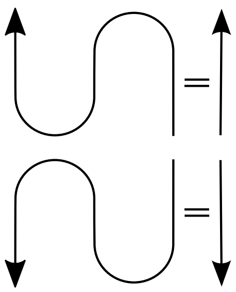


Figure 2.6: The zigzag identities simply mean that one can straighten strands as depicted.

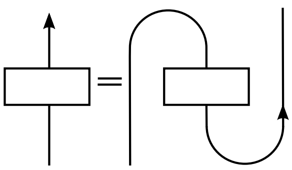


Figure 2.7: If a category $\mathscr{C}$ has duality, then duality defines a endofunctor on $\mathscr{C}$ with the dual of a map $f : X \to Y$ being the map $f^* : Y^* \to X^*$ defined by composing f with evaluation and coevaluation maps as depicted in this figure.

Definition 2.2.7 [Tur94]**.** Let $\mathscr{C}$ be a braided monoidal k–linear category with a braiding B. A *twist* in $\mathscr{C}$ is a k–linear natural isomorphism which on the component $a \in \mathscr{C}$ is $\theta_a : a \to a$ and satisfies

$$\theta_{a \otimes b} = B_{b,a} B_{a,b} (\theta_a \otimes \theta_b)$$

for all $a, b \in \mathscr{C}$. A braided monoidal category with duality and a twist is a *ribbon category* if the twist and duality are compatible, that is

$$(\theta_a \otimes \mathrm{Id}_{a^*})\eta_a = (\mathrm{Id}_a \otimes \theta_{a^*})\epsilon_a$$

for all $a \in \mathscr{C}$.

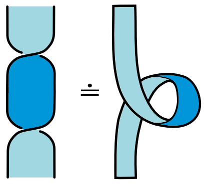


Figure 2.8: The graphic calculus of ribbon categories can be given by thickening the strands to ribbons (framed strands). The twist is just a twist of the ribbon. Alternatively, one can represent the ribbon category using just the cores of the bands if one represents the twists as loops. We shall defined this graphical calculus formally in Section 4.2.1 as it important in the definition of a Skein category.

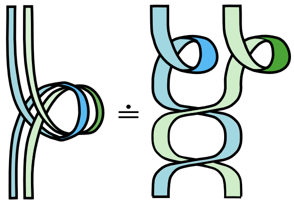


Figure 2.9: This figure shows the compatibility relation of the twist with the braiding.

2.2.1 Monoidal Structure of $\mathbf{Cat}_k$ and $\mathbf{Pr}$

The $(2,1)$–category $\mathbf{Cat}_k$ is a strict monoidal category with the categorical product $\times$ as monoidal product:

1. The product $\mathscr{C} \times \mathscr{D}$ has as objects tuples (m, n) where $m \in \mathscr{C}$ and $n \in \mathscr{D}$ and as morphisms tuples (f, g) where $f : m \to m'$ is a morphism in $\mathscr{C}$ and $g : n \to n'$ is a morphism in $\mathscr{D}$.

2. The monoidal unit $1_{\mathbf{Cat}}$ is the category $\mathbf{Pt}$ with a single object and a single morphism which is the identity morphism on this object

The $(2,1)$–category $\mathbf{Pr}$ is also a strict monoidal category but the monoidal product $\boxtimes$ is given by the Kelly–Deligne tensor product†.

Definition 2.2.8. The *Kelly–Deligne tensor product* of $\mathscr{A}, \mathscr{B} \in \mathbf{Pr}$ is a category $\mathscr{A} \boxtimes \mathscr{B} \in \mathbf{Pr}$ together with a bilinear functor $S : \mathscr{A} \times \mathscr{B} \to \mathscr{A} \boxtimes \mathscr{B}$ which is cocontinuous in each variable separately and defines an equivalence of categories

$$\mathbf{Cocont}(\mathscr{A} \boxtimes \mathscr{B}, \mathscr{C}) \simeq \mathbf{Cocont}(\mathscr{A}, \mathscr{B}; \mathscr{C}) \cong \mathbf{Cocont}(\mathscr{A}, \mathbf{Cocont}(\mathscr{B}, \mathscr{C}))$$

for all $\mathscr{C} \in \mathbf{Pr}$ given by composing functors with S: $\mathbf{Cocont}(\mathscr{A} \boxtimes \mathscr{B}, \mathscr{C})$ is the category of cocontinuous functors $\mathscr{A} \boxtimes \mathscr{B} \to \mathscr{C}$ and $\mathbf{Cocont}(\mathscr{A}, \mathscr{B}; \mathscr{C})$ is the category of bilinear functors $\mathscr{A} \times \mathscr{B} \to \mathscr{A} \boxtimes \mathscr{B}$ which are cocontinuous in each variable separately.

Remark 2.2.9. Kelly [Kel82] proved the existence of $\mathscr{A} \boxtimes \mathscr{B}$ for categories $\mathscr{A}, \mathscr{B} \in \mathbf{Rex}$, the $(2,1)$–category of essentially small, finitely cocomplete categories with right exact functors as 1–morphisms and natural isomorphisms as 2–morphisms. Franco in [LF13] shows that for abelian categories $\mathscr{A}, \mathscr{B}$, this tensor product $\mathscr{A} \boxtimes \mathscr{B}$ is the Deligne tensor product of abelian categories [Del90] when the Deligne tensor product exists; hence, the name Kelly-Deligne tensor product. For the existence of the Kelly-Deligne tensor product in $\mathbf{Pr}$ see [RG17, Section 2.4.1] and the references therewithin.

†The monoidal unit of $\mathbf{Pr}^{\boxtimes}$ is $k\mathbf{Mod}$.

2.3 Factorisation Homology

In this section we shall define factorisation homology. In the remainder of this thesis we shall only consider factorisation homologies of surfaces, that is fix $n = 2$, and we shall assume $\mathscr{C}^{\otimes} = \mathbf{Pr}^{\boxtimes}$ or $\mathbf{Cat}_k$: in Chapter 3 we use $\mathbf{Pr}^{\boxtimes}$ and in Chapter 4 we use both. General introductory references for factorisation homology include Ginot [Gin15] and Ayala and Francis [AF15, AF19].

Definition 2.3.1. A smooth manifold M is *finitary* if it has a finite open cover $\mathcal{U}$ such that if $\{ U_i \}$ is a subset of $\mathcal{U}$ then intersection $\cap_i U_i$ is either empty or diffeomorphic to $\mathbb{R}^n$.

Remark 2.3.2. Manifolds and surfaces are assumes throughout this thesis to be finitary and smooth.

Definition 2.3.3. Let X and Y be smooth framed manifolds and let $\mathbf{Emb}(X, Y)$ denote the ∞–groupoid of the topological space of smooth embeddings of X into Y which respect the framings with the smooth compact open topology, i.e the objects of $\mathbf{Emb}(X, Y)$ are smooth framed embeddings, the 1–morphisms are isotopies, the 2–morphisms are homotopies between the 1–morphisms and so on.

Definition 2.3.4. Let $\mathbf{Mfld}^n_{\mathrm{fr}}$ be the symmetric monodial ∞–category whose objects are framed manifolds, whose Hom–space of morphisms between manifolds X and Y is the ∞–groupoid $\mathbf{Emb}(X, Y)$, and whose symmetric monodial structure is given by disjoint union.

Definition 2.3.5. Let $\mathbf{Disc}^n$ be the full subcategory of $\mathbf{Mfld}^n_{\mathrm{fr}}$ of disjoint unions of $\mathbb{R}^n$. Denote the inclusion functor by $I : \mathbf{Disc}^n \to \mathbf{Mfld}^n_{\mathrm{fr}}$.

Definition 2.3.6. An *E_n–algebra* is a symmetric monoidal functor $F : \mathbf{Disc}^n \to \mathscr{C}^{\otimes}$ where $\mathscr{C}^{\otimes}$ is a symmetric monoidal ∞–category. As F is determined on objects by its value of a single disc, we define $\mathscr{E} := F(\mathbb{R}^n)$, and we use $\mathscr{E}$ to refer to the associated E_n–algebra.

Definition 2.3.7 [AF15]**.** A symmetric monoidal ∞-category $\mathscr{C}$ is $\otimes$–presentable if

1. $\mathscr{C}$ is locally presentable with respect to an infinite cardinal κ and
2. the monoidal structure distributes over small colimits i.e. the functor $C \otimes \_ : \mathscr{C} \to \mathscr{C}$ carries colimit diagrams to colimit diagrams.

Remark 2.3.8. Both $\mathbf{Pr}^{\boxtimes}$ and $\mathbf{Cat}_k^{\times}$ are $\otimes$–presentable [BZBJ18a, KL01, Kel05].

Definition 2.3.9. Let $\mathscr{C}^{\otimes}$ be a $\otimes$–presentable[†] symmetric monoidal ∞–category and let $F : \mathbf{Disc}^n \to \mathscr{C}^{\otimes}$ be an E_n–algebra with $\mathscr{E} := F(\mathbb{R}^n)$. The left Kan extension of the diagram

$$\begin{tikzcd} \mathbf{Disc}^n \arrow[r, "F"] \arrow[d, hook, "I"'] & \mathscr{C}^{\otimes} \\ \mathbf{Mfld}^n_{\mathrm{fr}} \arrow[ur, dashed, "\int_{\_} \mathscr{E}"'] & \end{tikzcd}$$

is called the[*] *factorisation homology* with coefficients in $\mathscr{E}$; its image on the manifold Σ is called the factorisation homology of Σ over $\mathscr{E}$ and is denoted $\int_{\Sigma} \mathscr{E}$.

[†]Slightly weaker conditions than $\otimes$–presentable are possible see [AF15]

[*]As factorisation homology is defined via a universal construction we have uniqueness up to a contractible space of isomorphisms.

2.3.1 Excision

Factorisation homology like classical homology satisfies an excision property: the factorisation homology of a cylinder gluing of two manifolds can be obtained from the factorisation homology of the original manifolds by tensoring relative to the submanifold glued along; hence, factorisation homology is determined locally.

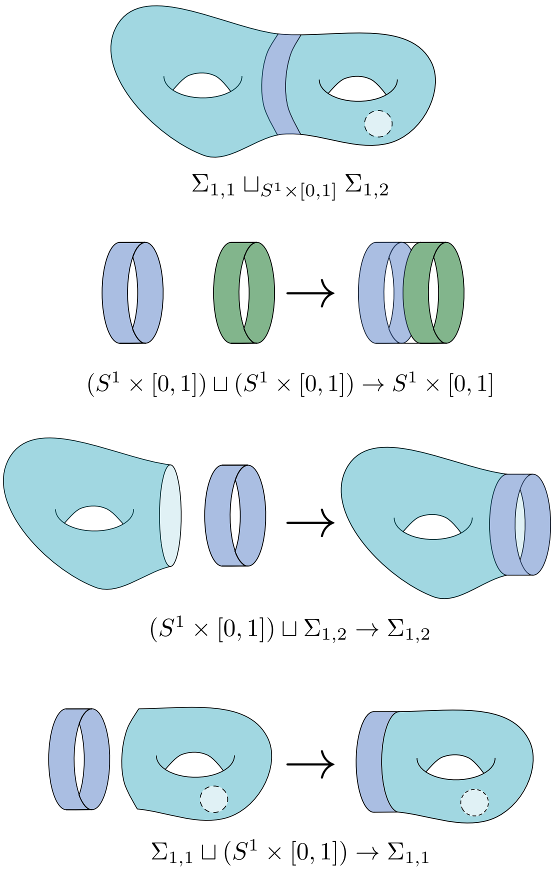


Figure 2.10: An example of the maps which induce the monoidal and module structures of the factorisation homologies.

When $\Sigma = C \times [0,1]$ for some $(n-1)$–dimensional manifold C, the factorisation homology $\int_{C\times[0,1]} \mathscr{E}$ can be equipped with a monoidal structure induced by the embedding

$$(C \times [0,1]) \sqcup (C \times [0,1]) \hookrightarrow C \times [0,1]$$

which retracts both copies of $C \times [0,1]$ in the second coordinate and includes them into another copy of $C \times [0,1]$.

Let $\Sigma = M \sqcup_{C\times[0,1]} N$ be the collar gluing of the n–dimensional manifolds M and N along $C \times [0,1]$. The embeddings

$$M \sqcup (C \times [0,1]) \hookrightarrow M$$
$$(C \times [0,1]) \times N \hookrightarrow N$$

induces a right $\int_{C\times[0,1]} \mathscr{E}$–module structure on $\int_M \mathscr{E}$ and a left $\int_{C\times[0,1]} \mathscr{E}$–module structure on $\int_N \mathscr{E}$. In other words, $\int_{C\times[0,1]} \mathscr{E}$ is an algebra object in $\mathscr{C}^{\otimes}$, and $\int_M \mathscr{E}$ and $\int_N \mathscr{E}$ are right and left modules over this algebra object.

Definition 2.3.10. Let $\mathscr{C}^{\otimes}$ be a $\otimes$-presentable symmetric monoidal ∞–category. Let $\mathscr{A}$ be an algebra object in $\mathscr{C}^{\otimes}$, and let $\mathscr{M}$ and $\mathscr{N} \in \mathscr{C}^{\otimes}$ be right and left modules respectively over the algebra object $\mathscr{A}$. The *relative tensor product* $\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}$ is the colimit of the 2-sided bar construction

$$\dots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathscr{M} \otimes \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{N} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathscr{M} \otimes \mathscr{A} \otimes \mathscr{N} \;\substack{\longrightarrow\\ \longrightarrow}\; \mathscr{M} \otimes \mathscr{N}$$

Remark 2.3.11. If $\mathscr{C}^{\otimes} = \mathbf{Pr}^{\boxtimes}$ or $\mathbf{Cat}_k$ then this 2–sided bar construction strictifies after the second step as they are 2-categories. We shall show in Section 4.1 that this colimit is given by the relative tensor product of Tambara (see Definition 4.1.7) which is called the relative Kelly–Deligne tensor product in [BZBJ18a].

Theorem 2.3.12 [AF15]**.** *Let $\Sigma = M \sqcup_{C\times[0,1]} N$ be the collar gluing of the n–dimensional manifolds M and N along $C \times [0,1]$ where C is a $(n-1)$–dimensional manifold. There is an equivalence of categories*

$$\int_{M\sqcup_{C\times[0,1]}} \mathscr{E} \simeq \int_M \mathscr{E} \otimes_{\int_{C\times[0,1]}\mathscr{E}} \int_N \mathscr{E}.$$

2.3.2 Other Properties of Factorisation Homology

As $\emptyset$ is the identity for the monoidal product $\sqcup$ in $\mathbf{Mfld}_{\mathrm{fr}}$,

$$\int_{\emptyset} \mathscr{E} \simeq 1_{\mathscr{C}^{\otimes}}.$$

We can embed the empty manifold into any manifold, and this embedding $\emptyset \to \Sigma$ induces a morphism $1_{\mathscr{C}^{\otimes}} \simeq \int_{\emptyset} \mathscr{E} \to \int_{\Sigma} \mathscr{E}$, giving a pointed structure to factorisation homology.

Theorem 2.3.13 [BZBJ18a, AF15, AFT17]**.** *Let $\mathscr{E}$ be an E_2–algebra in $\mathscr{C}^{\otimes}$. The functor $\int_{-} \mathscr{E}$ is characterised by the following properties:*

1. *If U is contractible then then is an equivalence in $\mathscr{C}^{\otimes}$*
$$\int_U \mathscr{E} \simeq \mathscr{E};$$
2. *If $M \cong C \times [0,1]$ for some 1–manifold with corners C then the inclusion of intervals inside a larger interval induces a canonical E_1–structure on $\int_M \mathscr{E}$.*
3. *$\int_{-} \mathscr{E}$ satisfies excision (see Theorem 2.3.12).*

2.4 Reduction Systems and the Diamond Lemma

Both the universal enveloping algebra of a Lie algebra $\mathcal{U}(\mathfrak{g})$ and its quantum group $\mathcal{U}_q(\mathfrak{g})$ have a Poincare–Birkhoff–Witt basis (PBW–basis). In the case of $\mathcal{U}(\mathfrak{g})$ this means that if $x_1, \dots, x_l$ is an ordered basis of $\mathfrak{g}$ then $\mathcal{U}(\mathfrak{g})$ has a vector space basis given by the monomials

$$y_1^{k_1} y_2^{k_2} \dots y_l^{k_l}$$

where $k_i \in \mathbb{N}_0$ and $x_i \mapsto y_i$ via the map $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$. In the case of $\mathcal{U}_q(\mathfrak{g})$ this means that $\mathcal{U}_q(\mathfrak{g})$ has a vector space basis given by the monomials

$$(X_1^+)^{a_1} \dots (X_n^+)^{a_n} K_1^{b_1} \dots K_n^{b_n} (X_1^-)^{c_1} \dots (X_n^-)^{c_n}$$

where $a_i, c_i \geq 0$ and $b_i \in \mathbb{Z}$.

In this section we recall the definitions and results needed to define and prove the existence of such bases. We will use these results in Section 3.2 and Section 3.5 to provide PBW–bases for the algebra objects and invariant algebras of the factorisation homology of the four–punctured sphere and punctured torus with coefficients in $\mathbf{Rep}_q(\mathrm{SL}_2)$. The definitions given in this section can be found [Ber78] except those relating to the reduced degree which can be found in [Cas17], and the main result is the Diamond lemma for rings proven by Bergman in [Ber78]. Let k be a commutative ring with multiplicative identity and X be an alphabet (a set of symbols from which we form words).

Definition 2.4.1. A *reduction system* S consists of term rewriting rules $\sigma : W_\sigma \mapsto f_\sigma$ where $W_\sigma \in \langle X \rangle$ is a word in the alphabet X and $f_\sigma \in k\langle X \rangle$ is a linear combination of words. A *σ–reduction* $r_\sigma(T)$ of an expression $T \in k\langle X \rangle$ is formed by replacing an instance of W_σ in T with f_σ. For example, if $X = \langle a, b \rangle$ and $S = \{\, \sigma : ab \mapsto ba \,\}$ then $r_\sigma(T) = aba + a$ is a σ–reduction of $T = aab + a$. A *reduction* is a σ–reduction for some $\sigma \in S$.

Definition 2.4.2. The five–tuple (σ, τ, A, B, C) with $\sigma, \tau \in S$ and $A, B, C \in \langle X \rangle$ is an *overlap ambiguity* if $W_\sigma = AB$ and $W_\tau = BC$ and an *inclusion ambiguity* if $W_\sigma = B$ and $W_\tau = ABC$. These ambiguities are *resolvable* if reducing ABC by starting with a σ–reduction gives the same result as starting with a τ–reduction. For example if $S = \{\, \sigma : ab \mapsto ba, \tau : ba \mapsto a \,\}$ then (σ, τ, a, b, a) is an overlap ambiguity which is resolvable as $aba \xmapsto{r_\sigma} ba^2 \xmapsto{r_\tau} a^2$ gives the same expression as $aba \xmapsto{r_\tau} a^2$.

Definition 2.4.3. A *semigroup partial ordering* $\leq$ on $\langle X \rangle$ is a partial order such that $B \leq B'$ implies that $ABC \leq AB'C$ for all words A, B, B', C; it is *compatible with the reduction system* S if for all $\sigma \in S$ the monomials in f_σ are less than W_σ.

Definition 2.4.4. A reduction system S satisfies the *descending chain condition* or is *terminating* if for any expression $T \in k\langle X \rangle$ any sequence of reductions terminates in a finite number of reductions with an irreducible expression.

Lemma 2.4.5 The Diamond Lemma [Ber78]**.** *Let S be a reduction system for $k\langle X \rangle$ and let $\leq$ be a semigroup partial ordering on $\langle X \rangle$ compatible with the reduction system S with the descending chain condition. The following are equivalent:*

1. *All ambiguities in S are resolvable (S is* locally confluent*);*

2. *Every element $a \in k\langle X \rangle$ can be reduced in a finite number of reductions to a unique expression $r_S(a)$ (S is* confluent*);*

3. *The algebra $R = k\langle X \rangle / I$, where I is the two sided ideal of $k\langle X \rangle$ generated by the elements $(W_\sigma - f_\sigma)$, can be identified with the k–algebra $k\langle X \rangle_{\mathrm{irr}}$ spanned by the S–irreducible monomials of $\langle X \rangle$ with multiplication given by $a \cdot b = r_S(ab)$. These S–irreducible monomials are called a Poincare–Birkhoff–Witt–basis of R.*

Remark 2.4.6. Bergman's Diamond Lemma is an application to ring theory of the Diamond Lemma for abstract rewriting systems. An *abstract rewriting system* is a set A together with a binary relation $\to$ on A called the *reduction relation* or *rewrite relation.*

1. It is *terminating* if there are no infinite chains $a_0 \to a_1 \to a_2 \to \dots$.

2. It is *locally confluent* if for all $y \leftarrow x \to z$ there exists an element $y \downarrow z \in A$ such that there are paths $y \to \cdots \to (y \downarrow z)$ and $z \to \cdots \to (y \downarrow z)$.

3. It is *confluent* if for all $y \leftarrow \dots \leftarrow x \to \dots \to z$ there exists an element $y \downarrow z \in A$ such that there are paths $y \to \cdots \to (y \downarrow z)$ and $z \to \cdots \to (y \downarrow z)$. In a terminating confluent abstract rewriting system an element $a \in A$ will always reduce to a unique reduced expression regardless of the order of the reductions used.

The Diamond Lemma (or Newman's lemma) for abstract rewriting systems states that a terminating abstract rewriting system is confluent if and only if it is locally confluent.

$$
\begin{tikzcd}
& & \mathrm{x} \arrow[dl] \arrow[dr] & & \\
& \mathrm{b} \arrow[dl, dotted] \arrow[dr, dotted] & & \mathrm{c} \arrow[dl, dotted] \arrow[dr, dotted] & \\
\mathrm{a} \arrow[drr, dotted] & & \mathrm{b} \downarrow c & & \mathrm{d} \arrow[dll, dotted] \\
& & \mathrm{a} \downarrow d & &
\end{tikzcd}
$$

Figure 2.11: If the abstract term rewriting system is locally confluent there exists $b \downarrow d$ forming a small diamond shape. If it confluent there exists $a \downarrow d$ forming a larger diamond shape. The Diamond lemma is proven by patching together the small diamonds to give the larger diamonds and inducting on path length, hence the name.

In this thesis the semigroup partial ordering we shall use is ordering by *reduced degree*:

Definition 2.4.7. Give the letters of the finite alphabet X an ordering $x_1 \leq \cdots \leq x_N$. Any word W of length n can be written as $W = x_{i_1} \dots x_{i_n}$ where $x_{i_j} \in X$. An *inversion* of W is a pair $k \leq l$ with $x_{i_k} \geq x_{i_l}$ i.e. a pair with letters in the incorrect order. The number of inversions of W is denoted $|W|$.

Definition 2.4.8. Any expression T can be written as a linear combination of words $T = \sum c_l W_l$. Define $\rho_n(T) := \sum_{\text{length}(W_l)=n, c_l \neq 0} |W_l|$. The *reduced degree of* T is the largest n such that $\rho_n(T) \neq 0$.

Definition 2.4.9. Under the *reduced degree ordering*, $T \leq S$ if

1. The reduced degree of T is less than the reduced degree of S, or

2. The reduced degree of T and S are equal, but $\rho_n(T) \leq \rho_n(S)$ for maximal nonzero n.

Chapter 3

Quantum Character Varieties via Factorisation Homology

In this chapter we wish to consider the factorisation homology of the four–punctured sphere $\Sigma_{0,4}$ and punctured torus $\Sigma_{1,1}$ with coefficients in the category $\mathbf{Rep}_q(\mathrm{SL}_2)$ of integrable representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. Throughout this chapter we set $k = \mathbb{C}$ and assume $q \in \mathbb{C}$ is not a root of unity.

3.1 Factorisation Homology of Quantum Groups

The first step is to describe $\int_\Sigma \mathbf{Rep}_q(\mathrm{SL}_2)$ for $\Sigma = \Sigma_{0,4}$ or $\Sigma_{1,1}$ as a category of modules of an algebra and give a presentation for this algebra which is a straightforward application of the work of Ben-Zvi, Brochier and Jordan [BZBJ18a]. In this section we shall define $\mathbf{Rep}_q(G)$ and briefly outline the relevant results from [BZBJ18a].

3.1.1 Category of Integrable Representations of Quantum Groups

Let G be a connected Lie group such that $\mathrm{Lie}(G) = \mathfrak{g}$ is a finite–dimensional complex semisimple Lie algebra. Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g}$, $\langle \cdot, \cdot \rangle$ denote the Killing form, and $\Pi = \{\alpha_1, \dots, \alpha_n\}$ denote the simple roots.

Definition 3.1.1. A representation of $\mathcal{U}(\mathfrak{g})$ is *integrable* if it is the differential of a representation of G.

Remark 3.1.2. As SL_2 is simply–connected every representation of $\mathfrak{sl}_2 = \mathrm{Lie}(\mathrm{SL}_2)$ is integrable.

Proposition 3.1.3 [CP94]**.** *Every finite–dimensional $\mathcal{U}_q(\mathfrak{g})$–module is semisimple and the decomposition corresponds to the decomposition of finite–dimensional $\mathfrak{g}$–modules. The simple modules are characterised by their highest weights. The highest weights of $\mathcal{U}_q(\mathfrak{g})$ are*

$$\omega = \sigma(\alpha_i) q^{\langle \alpha_i, \lambda \rangle}$$

for any homomorphism $\sigma : \mathbb{Z}\Pi \to \pm 1$ and highest weight λ of $\mathfrak{g}$.

Definition 3.1.4. The finite–dimensional $\mathcal{U}_q(\mathfrak{g})$–module $V_1 \oplus \cdots \oplus V_n$ is of *type 1* if the highest weight of each simple module V_j has the form $q^{\langle \alpha_i, \lambda_j \rangle}$ for some highest weight λ_j of $\mathfrak{g}$ i.e $\sigma_j = 1$.

Corollary 3.1.5. *The finite–dimensional $\mathcal{U}_q(\mathfrak{g})$–modules of type* 1 *correspond to the finite–dimensional* $\mathfrak{g}$*–modules. Its* category of finite–dimensional representations $\mathbf{Rep}_q^{\mathrm{fd}}(G)$ *is the category with objects the finite–dimensional integrable modules of* $\mathcal{U}_q(\mathfrak{g})$ *of Type* 1 *and morphisms being module homomorphisms.*

Definition 3.1.6. Let G be a connected Lie group such that $\mathrm{Lie}(G) = \mathfrak{g}$ is a finite–dimensional complex semisimple Lie algebra. The *finite–dimensional integrable* representations of $\mathcal{U}_q(\mathfrak{g})$ are the finite–dimensional, type 1 $\mathcal{U}_q(\mathfrak{g})$–modules which correspond to integrable $\mathfrak{g}$–modules.

We are now in a position to define $\mathbf{Rep}_q^{\mathrm{fd}}(G)$ and $\mathbf{Rep}_q(G)$.

Definition 3.1.7. Let G be a connected Lie group with semisimple Lie algebra $\mathfrak{g}$. The *category of finite–dimensional integrable representations* $\mathbf{Rep}_q^{\mathrm{fd}}(G)$ is the category with objects the finite–dimensional integrable $\mathcal{U}_q(\mathfrak{g})$–modules and morphisms being module homomorphisms.

Definition 3.1.8. Let G be a connected Lie group with semisimple Lie algebra $\mathfrak{g}$. The *category of integrable representations* $\mathbf{Rep}_q(G)$ is the category with objects being possibly infinite direct sums of simple finite–dimensional integrable $\mathcal{U}_q(\mathfrak{g})$–modules and morphisms being module homomorphisms†.

We shall now equip $\mathbf{Rep}_q^{\mathrm{fd}}(G)$ with the structures of a ribbon category; $\mathbf{Rep}_q(G)$ inherits its ribbon structure from $\mathbf{Rep}_q^{\mathrm{fd}}(G)$. For more details see [CP94, ST09, KT09].

I. The monoidal product

$$\otimes : \mathbf{Rep}_q^{\mathrm{fd}}(G) \times \mathbf{Rep}_q^{\mathrm{fd}}(G) \to \mathbf{Rep}_q^{\mathrm{fd}}(G)$$

is defined as follows: if $V, W \in \mathbf{Rep}_q^{\mathrm{fd}}(G)$ then $V \otimes W$ is the vector space $V \otimes_{\mathbb{C}} W$ equipped with $\mathcal{U}_q(\mathfrak{g})$ action defined by $g \cdot (V \otimes W) = (\Delta(g)_1 \cdot V, \Delta(g)_2 \cdot W)$.

II. The monoidal category $\mathbf{Rep}_q^{\mathrm{fd}}(G)$ has duality. Let S denote the antipode of $\mathcal{U}_q(\mathfrak{g})$. The dual of $V \in \mathbf{Rep}_q^{\mathrm{fd}}(G)$ is the dual vector space $V^* = \mathrm{Hom}(V, \mathbb{C})$ with $\mathcal{U}_q(\mathfrak{g})$ action defined by $g \cdot f(v) = f(S(g)v)$ for $g \in \mathcal{U}_q(\mathfrak{g})$, $v \in V$ and $f \in V^*$.

III. We define $R := (X^{-1} \otimes X^{-1})\Delta(X)$ where $X := J\tilde{\omega}_{h,0}$: $\tilde{\omega}_{h,0}$ is the quantum Weyl element corresponding to the longest element ω_0 of the Weyl group of $\mathfrak{g}$, and J is the operator which acts on finite–dimensional representations of $\mathcal{U}_q(\mathfrak{g})$ by multiplying each vector of weight μ by $q^{\frac{1}{2}\langle \mu,\mu\rangle + \langle \mu,\rho\rangle}$. A braiding of $\mathbf{Rep}_q^{\mathrm{fd}}(G)$ is given by $c^R_{V,W}(V \otimes W) = \tau_{V,W}(\mathscr{R}_h(V \otimes W))$.

IV. The twist θ is defined as follows: θ acts on the irreducible representation V_λ of highest weight λ as the constant $q^{-\langle \lambda,\lambda\rangle - 2\langle \lambda,\rho\rangle}$ where $\rho \in \mathfrak{h}$ such that $\langle \alpha_i, \rho\rangle = d_i$ for all i.

Remark 3.1.9. Morally R should be considered as an R–matrix of $\mathcal{U}_q(\mathfrak{g})$ with

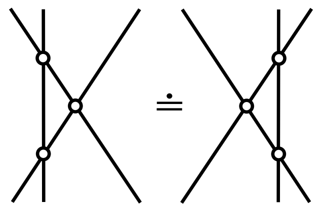
$$J := \exp\left[h \left(\frac{1}{2} \sum_{i,j} (B^{-1})_{ij} H_i \otimes H_j + H_\rho \right) \right];$$

however, R is not an element of $\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$, so isn't. It is, however, an R–matrix of the non-specialised quantum group $\mathcal{U}_h(\mathfrak{g})$.

†Note that $\mathbf{Rep}_q(G)$ is the ind-completion of the category of finite dimensional modules

Universal R–matrices generate solutions to the Yang–Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$



The Yang–Baxter equation has a physical interpretation as follows. Suppose one is modelling scattering of identical particles, and one assumes such scattering does not create or destroy particles. One can associate to each particle in the system a vector space V. The elastic collision of two particles is modelled by transforming the initial state $V \otimes V$ by applying the R–matrix. Now one wishes to consider the collision of three particles. Due to relativistic effects whether the three particles collided simultaneously or pairwise, and in which order, depends on the observer, so the collision of three particles can be modelled as a sequence of of R–matrix applications which model the pairwise collisions. The quantum Yang–Baxter equation is a consistency relation which ensures that the two ways of resolving the collision pairwise give the same result.

3.1.2 Computing the Factorisation Homology for Punctured Surfaces

The factorisation homology $\int_\Sigma \mathscr{E}$ of the punctured surface Σ is an $\mathscr{E}$–module category.

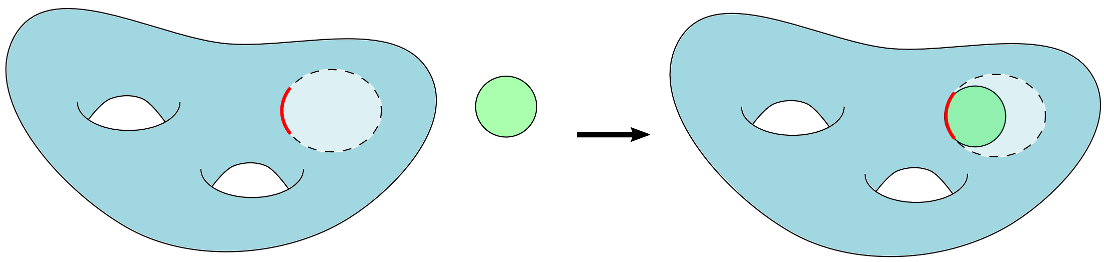


Figure 3.1: An illustration of the map $\Sigma \sqcup \mathbb{D} \to \Sigma$. The surface $\Sigma_{2,1}$ has a interval marked in red along its boundary along which the disc $\mathbb{D}$ is attached. The resultant surface is isotopic to $\Sigma_{2,1}$.

Choose a interval along its boundary†.

$$\Sigma \sqcup \mathscr{D} \to \Sigma,$$

which attaches the disc $\mathbb{D}$ to Σ along the marked interval, induces a $\int_{\mathbb{D}} \mathscr{E}$–module structure on $\int_\Sigma \mathscr{E}$. As $\int_{\mathbb{D}} \mathscr{E} \simeq \mathscr{E}$ in $\mathscr{C}^{\otimes}$, this means that $\int_\Sigma \mathscr{E}$ is a $\mathscr{E}$–module. Not only is $\int_\Sigma \mathscr{E}$ module category, but it is also the category of modules of an algebra.

Definition 3.1.10. Let $\mathscr{C}^{\boxtimes} = \mathbf{Rex}^{\boxtimes}$. The *distinguished object* $\mathscr{O}_{\mathscr{E},\Sigma}$ of a factorisation homology of Σ over $\mathscr{E}$ is the image of k under the pointing map $\mathrm{Vect}_k \to \int_\Sigma \mathscr{E}$.

Definition 3.1.11. The *algebra object* A_Σ of the factorisation homology of Σ* with coefficients in $\mathscr{E}$ is the internal endomorphism algebra of the distinguished object

$$A_\Sigma := \underline{\mathrm{End}}_{\mathscr{E}}(\mathscr{O}_{\mathscr{E},\Sigma}).$$

†The module structure depends on the choice of marking.
*The algebra object is dependent on the choice marking of Σ

This is called the *moduli algebra* of Σ in [BZBJ18a].

Proposition 3.1.12. *[BZBJ18a] Let Σ be a punctured surface, and $\mathscr{E}$ be a rigid braided tensor category, for example $\mathbf{Rep}_q(G)$ where G is a reductive algebraic group. We have an equivalence of categories*

$$\int_\Sigma \mathscr{E} \simeq A_\Sigma\text{-}\underline{\mathrm{mod}}_{\mathbf{Rep}_q(G)},$$

where A_Σ is the algebra object of the factorisation homology.

Remark 3.1.13. Note that as the factorisation homology is equivalent to a category of modules of an algebra, it is an abelian category.

There is a combinatorial description of A_Σ in terms of the gluing pattern of the surface.

Definition 3.1.14. A *gluing pattern* is a bijection

$$P : \{1, 1', \dots, n, n'\} \to \{1, 2, \dots, 2n-1, 2n\}$$

such that $P(i) < P(i')$ for all $i = 1, \dots, n$.

A gluing pattern P determines a marked surface $\Sigma(P)$ by gluing together a disc and n handles $H_i \cong [0,1]^2$ as follows: mark the disc with $2n+1$ boundary intervals labelled $1, \dots, 2n+1$; for each handle H_i mark two intervals i and i' on the boundary; glue the handles to the disc by identifying the interval i with the interval $P(i)$ and the interval i' with the interval $P(i')$ for all $i = 1, \dots, n$. The final interval $2n+1$ on the boundary of the disc gives $\Sigma(P)$ a marking.

Definition 3.1.15. The handles H_i and H_j, with $i < j$ are:

1. positively linked if $P(i) < P(j) < P(i') < P(j')$,
2. positively nested if $P(i) < P(j) < P(j') < P(i')$,
3. positively unlinked if $P(i) < P(i') < P(j) < P(j')$.

By relabelling the handles we can assume all handles are of the above forms.

Example 3.1.16. The four–punctured sphere has the simplest possible gluing pattern with three handles

$$\begin{aligned}&P : \{1, 1', 2, 2', 3, 3'\} \to \{1, 2, 3, 4, 5, 6\} :\\ &P(1) = 1, P(1') = 2, P(2) = 3, P(2') = 4, P(3) = 5, P(3') = 6.\end{aligned}$$

All three of its handles are positively unlinked.

Example 3.1.17. The punctured torus has the gluing pattern

$$P : \{1, 1', 2, 2'\} \to \{1, 2, 3, 4\} : P(1) = 1, P(1') = 3, P(2) = 2, P(2') = 4.$$

The handles H_1 and H_2 are positively linked.

Definition 3.1.18. Let $\mathscr{E} = \mathbf{Rep}_q(G)$ for a reductive algebraic Lie group G, $\mathscr{O}(\mathscr{E})$ is generated by elements of the form $v^i \otimes v_j \in V^* \otimes V$ for some representation $V \in \mathscr{E}$. We define the crossing morphism

$$K_{i,j} : \mathscr{O}(\mathscr{E})^{(i)} \otimes \mathscr{O}(\mathscr{E})^{(j)} \to \mathscr{O}(\mathscr{E})^{(i)} \otimes \mathscr{O}(\mathscr{E})^{(j)}$$

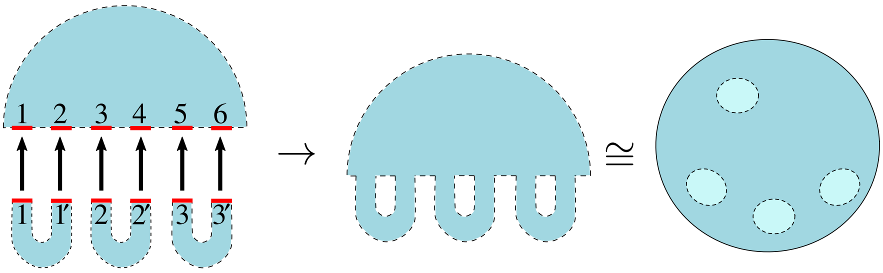


Figure 3.2: The gluing pattern of $\Sigma_{0,4}$.

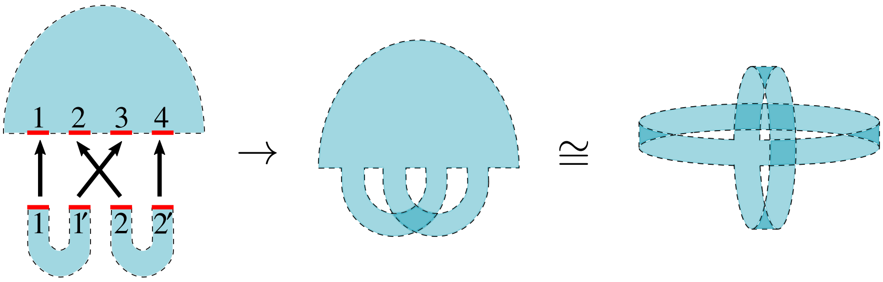
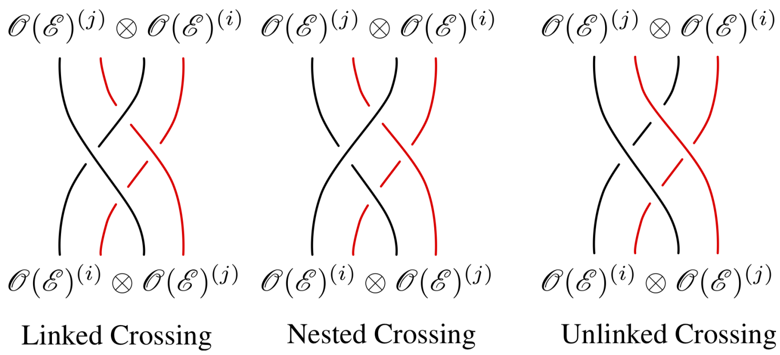


Figure 3.3: The gluing pattern of $\Sigma_{1,1}$.

using the braidings



where strand crossings are determined by the chosen R–matrix and antipode S of $\mathcal{U}_q(\mathfrak{g})$ are as follows:

$$\sigma_{V,W}(w \otimes v) = \tau_{V,W} \circ R(w \otimes v);$$
$$\sigma_{V^*,W}(w^* \otimes v) = \tau_{V^*,W} \circ (S \otimes id) \circ R(w^* \otimes v) = \tau_{V^*,W} \circ R^{-1}(w^* \otimes v);$$
$$\sigma_{V,W^*}(w \otimes v^*) = \tau_{V,W^*} \circ (id \otimes R) \circ R(w \otimes v^*);$$
$$\sigma_{V^*,W^*}(w^* \otimes v^*) = \tau_{V^*,W^*} \circ (S \otimes S) \circ R(w^* \otimes v^*) = \tau_{V^*,W^*} \circ R(w^* \otimes v^*).$$

As the crossing morphisms satisfy the Yang–Baxter equation, they can be used to extend the multiplication $m : \mathscr{O}(\mathscr{E}) \otimes \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E})$ to a associative multiplication map $m_n : \mathscr{O}(\mathscr{E})^{\otimes n} \otimes \mathscr{O}(\mathscr{E})^{\otimes n} \to \mathscr{O}(\mathscr{E})^{\otimes n}$ turning $\mathscr{O}(\mathscr{E})^{\otimes n}$ into an algebra [Leb13].

Proposition 3.1.19. *[BZBJ18a] Let $\Sigma(P)$ be a surface determined by a gluing pattern P and let $\mathscr{E} = \mathbf{Rep}_q(G)$ for a reductive algebraic Lie group G. Then $A_{\Sigma(P)}$ is isomorphic to the*

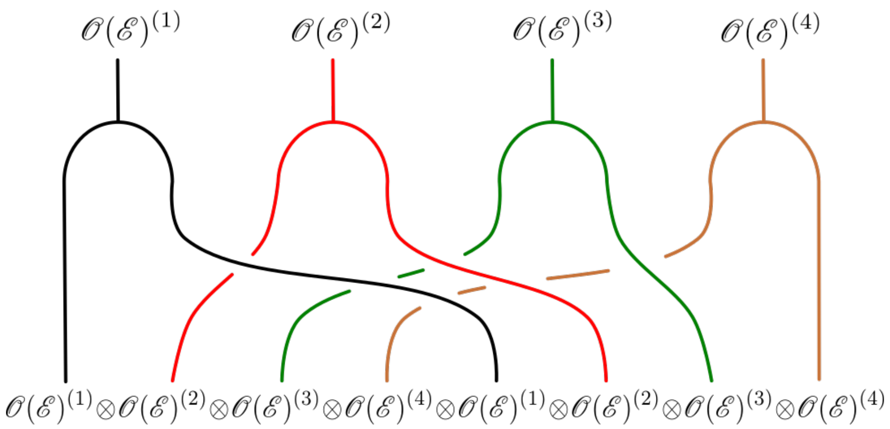


Figure 3.4: The multiplication map for $\mathscr{O}(\mathscr{E})^{\otimes 4}$ where the crossing of strands $\mathscr{O}(\mathscr{E})^{(i)}$ and $\mathscr{O}(\mathscr{E})^{(j)}$ is given by the braiding $K_{i,j}$

algebra

$$a_P = \mathscr{O}(\mathscr{E})^{(1)} \otimes \cdots \otimes \mathscr{O}(\mathscr{E})^{(n)},$$

where $\mathscr{O}(\mathscr{E})^{(i)}$ *is the reflection equation algebra of* $\mathcal{U}_q(\mathfrak{g})$*, and the crossing morphisms* $K_{i,j}$: $\mathscr{O}(\mathscr{E})^{(j)} \otimes \mathscr{O}(\mathscr{E})^{(i)} \to \mathscr{O}(\mathscr{E})^{(i)} \otimes \mathscr{O}(\mathscr{E})^{(j)}$ *where* i, j *are consecutive are given in Definition 3.1.18.*

3.2 The Factorisation Homology of the Four–Punctured Sphere and Punctured Torus over $\mathcal{U}_q(\mathfrak{sl}_2)$

Using Proposition 3.1.12 we have that the factorisation homology of the four–punctured sphere and punctured torus over $\mathcal{U}_q(\mathfrak{sl}_2)$ is A_Σ–$\underline{\mathrm{mod}}_{\mathbf{Rep}_q(G)}$ where A_Σ is the algebra object of the four–punctured sphere $\Sigma_{0,4}$ and punctured torus $\Sigma_{1,1}$ respectively. We shall use Proposition 3.1.19 to obtain presentations of $A_{\Sigma_{0,4}}$ and $A_{\Sigma_{1,1}}$. In order to do this, we require a presentation of the reflection equation algebra $\mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))$ and a description of $K_{i,j}$ in each case.

The R–matrix for $\mathcal{U}_q(\mathfrak{sl}_2)$ when evaluating on the standard representation of $\mathcal{U}_q(\mathfrak{sl}_2)$ is given by

$$R := \begin{pmatrix} R^{11}_{11} & R^{12}_{11} & R^{21}_{11} & R^{22}_{11} \\ R^{11}_{12} & R^{12}_{12} & R^{21}_{12} & R^{22}_{12} \\ R^{11}_{21} & R^{12}_{21} & R^{21}_{21} & R^{22}_{21} \\ R^{11}_{22} & R^{12}_{22} & R^{21}_{22} & R^{22}_{22} \end{pmatrix} := q^{\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & (q-q^{-1}) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

We shall also require

$$\tilde{R} := (\mathrm{Id} \otimes S)(R) = q^{-\frac{1}{2}} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & q^{-2}(q^{-1}-q) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}.$$

where S is the antipode of $\mathcal{U}_q(\mathfrak{sl}_2)$.

Definition 3.2.1. [BJ18] The reflection equation algebra $\mathscr{O}(Rep_q(\mathrm{SL}_2))$ is generated by the four elements

$$A = \begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix}$$

which satisfy the following:

1. The quantum determinant $\det_q(A) := a^1_1 a^2_2 - q^2 a^1_2 a^2_1 = 1$, and

2. The reflection equation $a^l_m a^p_r = \tilde{R}^{op}_{mk} (R^{-1})^{kl}_{ij} R^{sj}_{uv} R^{wu}_{or} a^i_s a^v_w$ where i, j, k, l, m, o, p, r, s, v, $w \in \{0, 1\}$†.

Or more explicitly the reflection equation algebra $\mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))$ has generators $a^1_1, a^1_2, a^2_1, a^2_2$ and relations

$$a^1_2 a^1_1 = a^1_1 a^1_2 + \left(1 - q^{-2}\right) a^1_2 a^2_2, \tag{3.1}$$
$$a^2_1 a^1_1 = a^1_1 a^2_1 - q^{-2} \left(1 - q^{-2}\right) a^2_1 a^2_2, \tag{3.2}$$
$$a^2_1 a^1_2 = a^1_2 a^2_1 + \left(1 - q^{-2}\right) \left(a^1_1 a^2_2 - a^2_2 a^2_2\right), \tag{3.3}$$
$$a^2_2 a^1_1 = a^1_1 a^2_2, \tag{3.4}$$
$$a^2_2 a^1_2 = q^2 a^1_2 a^2_2, \tag{3.5}$$
$$a^2_2 a^2_1 = q^{-2} a^2_1 a^2_2, \tag{3.6}$$
$$a^1_1 a^2_2 = 1 + q^2 a^1_2 a^2_1. \tag{3.7}$$

Definition 3.2.2. The braiding on $\mathbf{Rep}_q(\mathrm{SL}_2)$ for positively unlinked handles H_i and H_j is the map

$$K_{i,j} : \mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))^{(i)} \otimes \mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))^{(j)} \to \mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))^{(j)} \otimes \mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))^{(i)} :$$
$$K_{i,j}(y^e_f \otimes x^g_h) = \tilde{R}^{ig}_{fj} R^{ej}_{kl} R^{mn}_{ih} \left(R^{-1}\right)^{ko}_{pn} x^l_o \otimes y^p_m$$

where x^g_h and y^e_f are generators of $\mathbf{Rep}^{(i)}_q(\mathrm{SL}_2)$ and $\mathbf{Rep}^{(j)}_q(\mathrm{SL}_2)$ respectively.

Corollary 3.2.3. *The factorisation homology of the four–punctured sphere with coefficients in* $\mathbf{Rep}_q(\mathrm{SL}_2)$ *is* $\int_{\Sigma_{0,4}} \mathbf{Rep}_q(\mathrm{SL}_2) \simeq A_{\Sigma_{0,4}}\text{–}\underline{\mathrm{mod}}_{\mathbf{Rep}_q(\mathrm{SL}_2)}$ *where* $A_{\Sigma_{0,4}}$ *is an algebra with twelve generators organised into three matrices*

$$A := \begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix},\ B := \begin{pmatrix} b^1_1 & b^1_2 \\ b^2_1 & b^2_2 \end{pmatrix},\ C := \begin{pmatrix} c^1_1 & c^1_2 \\ c^2_1 & c^2_2 \end{pmatrix}$$

subject to the relations

$$x^1_1 x^2_2 = 1 + q^2 x^1_2 x^2_1 \qquad \textit{(determinant relation)} \tag{3.8}$$
$$y^l_m x^p_r = \tilde{R}^{op}_{mk} (R^{-1})^{kl}_{ij} R^{sj})_{uv} R^{wu}_{or} x^i_s y^v_w \qquad \textit{(reflection equation)} \tag{3.9}$$
$$y^e_f x^g_h = \tilde{R}^{ig}_{fj} R^{ej}_{kl} R^{mn}_{ih} (R^{-1})^{ko}_{pn} x^l_o y^p_m \qquad \textit{(crossing relation)} \tag{3.10}$$

†The reflection equation algebra is usually given as $R_{21} A_1 R A_2 = A_2 R_{21} A_1 R$ where $A_1 := A \otimes I$, $A_2 := I \otimes A$, and $R_{21} := \tau R \tau$, for example in [DM03] and [GPS08]. Our version is the tensor version rearranged using the relations $\sum (R^{-1})^{ij}_{kl} R^{kl}_{mn} = \delta^i_m \delta^j_n$ and $\sum \tilde{R}^{ij}_{kl} R^{ml}_{in} = \delta^m_k \delta^n_j$.

where $x \in \{a, b, c\}$, $e, f, g, h, i, j, k, l, m, n, o, p \in \{0, 1\}$,

$$R = q^{\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & (q - q^{-1}) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

is the standard quantum R–matrix for $\mathcal{U}_q(\mathfrak{sl}_2)$ *when evaluated on the standard representation of* $\mathcal{U}_q(\mathfrak{sl}_2)$ *and*

$$\tilde{R} = q^{-\frac{1}{2}} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & q^{-2}(q^{-1} - q) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}.$$

Definition 3.2.4. The braiding on $\mathbf{Rep}_q(\mathrm{SL}_2)$ for positively linked handles H_i and H_j is the map

$$\begin{aligned} &K_{i,j} : \mathbf{Rep}_q^{(i)}(\mathrm{SL}_2) \otimes \mathbf{Rep}_q^{(j)}(\mathrm{SL}_2) \to \mathbf{Rep}_q^{(j)}(\mathrm{SL}_2) \otimes \mathbf{Rep}_q^{(i)}(\mathrm{SL}_2) : \\ &K_{i,j}(y_h^g \otimes x_f^e) = \tilde{R}_{hj}^{ie} R_{kl}^{gj} R_{if}^{mn} \left(R^{-1}\right)_{pn}^{ko} x_o^l \otimes y_m^p \end{aligned}$$

where x_h^g and y_f^e are generators of $\mathbf{Rep}_q^{(i)}(\mathrm{SL}_2)$ and $\mathbf{Rep}_q^{(j)}(\mathrm{SL}_2)$ respectively.

Corollary 3.2.5. *The factorisation homology of the punctured torus with coefficients in* $\mathcal{U}_q(\mathfrak{sl}_2)$ *is* $\int_{\Sigma_{1,1}} \mathbf{Rep}_q(\mathrm{SL}_2) \simeq A_{\Sigma_{1,1}}\text{–}\underline{\mathrm{mod}}_{\mathbf{Rep}_q(\mathrm{SL}_2)}$ *where* $A_{\Sigma_{1,1}}$ *is an algebra with eight generators organised into two matrices*

$$A := \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}, \ B := \begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix}$$

subject to the relations

$$x_1^1 x_2^2 = 1 + q^2 x_2^1 x_1^2 \qquad \textit{(determinant relation)} \qquad (3.11)$$

$$y_m^l x_r^p = \tilde{R}_{mk}^{op} (R^{-1})_{ij}^{kl} R_{uv}^{sj} R_{or}^{wu} x_s^i y_w^v \qquad \textit{(reflection equation)} \qquad (3.12)$$

$$y_h^g x_f^e = \tilde{R}_{hj}^{ie} R_{kl}^{gj} R_{if}^{mn} \left(R^{-1}\right)_{pn}^{ko} x_o^l \otimes y_m^p \qquad \textit{(crossing relation)} \qquad (3.13)$$

where $x \in \{a, b, c\}$, $e, f, g, h, i, j, k, l, m, n, o, p \in \{0, 1\}$ *and the R–matrices are the same as in Corollary 3.2.3.*

3.2.1 Poincaré–Birkhoff–Witt bases for $A_{\Sigma_{0,4}}$ and $A_{\Sigma_{1,1}}$

We now construct a PBW basis for $\mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))$ which we shall use to construct PBW bases for $A_{\Sigma_{0,4}}$ and $A_{\Sigma_{1,1}}$.

Proposition 3.2.6. *The monomials*

$$\left\{ (a_1^1)^\alpha (a_2^1)^\beta (a_1^2)^\gamma (a_2^2)^\delta \;\middle|\; \alpha, \beta, \gamma, \delta \in \mathbb{N}_0, \ \beta \textit{ or } \gamma = 0 \right\}$$

are a PBW basis for the reflection equation algebra $\mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))$ *with respect to the ordering* $a_1^1 < a_2^1 < a_1^2 < a_2^2$.

Proof. The relations defining $\mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))$ can be re-expressed as the term rewriting system:

$$
\begin{aligned}
&\sigma_{1211} : a^1_2 a^1_1 \mapsto a^1_1 a^1_2 + \left(1 - q^{-2}\right) a^1_2 a^2_2,\\
&\sigma_{2111} : a^2_1 a^1_1 \mapsto a^1_1 a^2_1 - q^{-2}\left(1 - q^{-2}\right) a^2_1 a^2_2,\\
&\sigma_{2112} : a^2_1 a^1_2 \mapsto a^1_2 a^2_1 + \left(1 - q^{-2}\right)\left(a^1_1 a^2_2 - a^2_2 a^2_2\right),\\
&\sigma_{2211} : a^2_2 a^1_1 \mapsto a^1_1 a^2_2,\\
&\sigma_{2212} : a^2_2 a^1_2 \mapsto q^2 a^1_2 a^2_2,\\
&\sigma_{2221} : a^2_2 a^2_1 \mapsto q^{-2} a^2_1 a^2_2,\\
&\sigma_{1221} : a^1_2 a^2_1 \mapsto q^{-2} + q^{-2} a^1_1 a^2_2.
\end{aligned}
$$

The monomials listed in the statement of the result are the reduced monomials with respect to this term rewriting system; furthermore, there are no inclusion ambiguities, and the overlap ambiguities are

$$
\begin{aligned}
&(\sigma_{2112}, \sigma_{1211}, a^2_1, a^1_2, a^1_1), \quad (\sigma_{2212}, \sigma_{1211}, a^2_2, a^1_2, a^1_1),\\
&(\sigma_{2221}, \sigma_{2111}, a^2_2, a^2_1, a^1_1), \quad (\sigma_{2221}, \sigma_{2112}, a^2_2, a^2_1, a^1_2),\\
&(\sigma_{2112}, \sigma_{1221}, a^2_1, a^1_2, a^2_1), \quad (\sigma_{2212}, \sigma_{1221}, a^2_2, a^1_2, a^2_1),\\
&(\sigma_{1221}, \sigma_{2111}, a^1_2, a^2_1, a^1_1), \quad (\sigma_{1221}, \sigma_{2112}, a^1_2, a^2_1, a^1_2).
\end{aligned}
$$

We shall order $\mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))$ with respect to the reduced degree where we give the generators the ordering $a^1_1 < a^1_2 < a^2_1 < a^2_2$. This ordering is compatible with the given term rewriting systems and the rewriting will terminate, so if the ambiguities are resolvable then we can apply the Diamond lemma, and we are done. It can be checked by direct calculation that the ambiguities are resolvable[†]. For example for the first ambiguity we have that both

$$
\begin{aligned}
\left(a^2_1 a^1_2\right) a^1_1 &\overset{(\sigma_{2112})}{=} a^1_2\left(a^2_1 a^1_1\right) + \left(1 - q^{-2}\right)\left(a^1_1 a^2_2 a^1_1 - \left(a^2_2\right)^2 a^1_1\right)\\
&\overset{(\sigma_{2111},\, \sigma_{2211})}{=} \left(a^1_2 a^1_1\right) a^2_1 - q^{-2}\left(1 - q^{-2}\right) a^1_2 a^2_1 a^2_2\\
&\quad + \left(1 - q^{-2}\right)\left(\left(a^1_1\right)^2 a^2_2 - a^1_1\left(a^2_2\right)^2\right)\\
&\overset{(\sigma_{1211})}{=} a^1_1 a^1_2 a^2_1 + \left(1 - q^{-2}\right) a^1_2\left(a^2_2 a^2_1\right) - q^{-2}\left(1 - q^{-2}\right) a^1_2 a^2_1 a^2_2\\
&\quad + \left(1 - q^{-2}\right)\left(\left(a^1_1\right)^2 a^2_2 - a^1_1\left(a^2_2\right)^2\right)\\
&\overset{(\sigma_{2221})}{=} a^1_1 a^1_2 a^2_1 + q^{-2}\left(1 - q^{-2}\right) a^1_2 a^2_1 a^2_2 - q^{-2}\left(1 - q^{-2}\right) a^1_2 a^2_1 a^2_2\\
&\quad + \left(1 - q^{-2}\right)\left(\left(a^1_1\right)^2 a^2_2 - a^1_1\left(a^2_2\right)^2\right)\\
&= a^1_1 a^1_2 a^2_1 + \left(1 - q^{-2}\right)\left(\left(a^1_1\right)^2 a^2_2 - a^1_1\left(a^2_2\right)^2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
a^2_1\left(a^1_2 a^1_1\right) &\overset{(\sigma_{1211})}{=} \left(a^2_1 a^1_1\right) a^1_2 + \left(1 - q^{-2}\right) a^2_1 a^1_2 a^2_2\\
&\overset{(\sigma_{2111})}{=} a^1_1 a^2_1 a^1_2 - q^{-2}\left(1 - q^{-2}\right) a^2_1\left(a^2_2 a^1_2\right) + \left(1 - q^{-2}\right) a^2_1 a^1_2 a^2_2\\
&\overset{(\sigma_{2212})}{=} a^1_1 a^2_1 a^1_2 - \left(1 - q^{-2}\right) a^2_1 a^1_2 a^2_2 + \left(1 - q^{-2}\right) a^2_1 a^1_2 a^2_2
\end{aligned}
$$

[†]We used the computer algebra system MAGMA to check this and similar computations throughout this chapter.

$$
\begin{aligned}
&= && a_1^1\left(a_1^2 a_2^1\right) \\
&\stackrel{(\sigma_{2112})}{=} && a_1^1 a_2^1 a_1^2 + \left(1-q^{-2}\right)\left(\left(a_1^1\right)^2 a_2^2 - a_1^1\left(a_2^2\right)^2\right)
\end{aligned}
$$

give the same result, so the first ambiguity is resolvable. □

Proposition 3.2.7. *A PBW basis for $A_{\Sigma_{0,4}}$ is*

$$
\begin{aligned}
\{\, &(a_1^1)^{\alpha_1}(a_2^1)^{\beta_1}(a_1^2)^{\gamma_1}(a_2^2)^{\delta_1}(b_1^1)^{\alpha_2}(b_2^1)^{\beta_2}(b_1^2)^{\gamma_2}(b_2^2)^{\delta_2}(c_1^1)^{\alpha_3}(c_2^1)^{\beta_3}(c_1^2)^{\gamma_3}(c_2^2)^{\delta_3} \,| \\
&\qquad\qquad | \; \alpha_i,\beta_i,\gamma_i \in \mathbb{N}_0, \beta_i \text{ or } \gamma_i = 0 \,\}\,.
\end{aligned}
$$

Proof. By Proposition 3.2.6 we have a PBW basis

$$
\left\{ (a_1^1)^{\alpha}(a_2^1)^{\beta}(a_1^2)^{\gamma}(a_2^2)^{\delta} \;\middle|\; \alpha,\beta,\gamma,\delta \in \mathbb{N}_0,\; \beta \text{ or } \gamma = 0 \right\}
$$

for the reflection equation algebra $\mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))$. The algebra $A_{\Sigma_{0,4}}$ is the tensor product of three copies of $\mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))$; hence,

$$
\begin{aligned}
\{\, &(a_1^1)^{\alpha_1}(a_2^1)^{\beta_1}(a_1^2)^{\gamma_1}(a_2^2)^{\delta_1}(b_1^1)^{\alpha_2}(b_2^1)^{\beta_2}(b_1^2)^{\gamma_2}(b_2^2)^{\delta_2}(c_1^1)^{\alpha_3}(c_2^1)^{\beta_3}(c_1^2)^{\gamma_3}(c_2^2)^{\delta_3} \,| \\
&\qquad\qquad | \; \alpha_i,\beta_i,\gamma_i \in \mathbb{N}_0,\; \beta_i \text{ or } \gamma_i = 0 \,\}\,.
\end{aligned}
$$

is a PBW basis for it. □

Proposition 3.2.8. *A PBW basis for $A_{\Sigma_{1,1}}$ is*

$$
\left\{ (a_1^1)^{\alpha_1}(a_2^1)^{\beta_1}(a_1^2)^{\gamma_1}(a_2^2)^{\delta_1}(b_1^1)^{\alpha_2}(b_2^1)^{\beta_2}(b_1^2)^{\gamma_2}(b_2^2)^{\delta_2} \;\middle|\; \alpha_i,\beta_i,\gamma_i \in \mathbb{N}_0,\; \beta_i \text{ or } \gamma_i = 0 \right\}.
$$

Proof. Similar to above. □

We will need an alternative PBW basis for $A_{\Sigma_{0,4}}$ in Section 3.4, so we shall now give an alternative basis for $\mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))$, and then use it to give the alternative PBW basis for $A_{\Sigma_{0,4}}$.

Proposition 3.2.9. *The monomials*

$$
\left\{ (a_1^2)^{\alpha}(a_1^1)^{\beta}(a_2^2)^{\gamma}(a_2^1)^{\delta} \;\middle|\; \alpha,\beta,\gamma,\delta \in \mathbb{N}_0,\; \beta \text{ or } \gamma = 0 \right\}
$$

are a PBW basis for the reflection equation algebra $\mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))$ with respect to the ordering $a_1^2 < a_1^1 < a_2^2 < a_2^1$.

Proof. A term rewriting system for $\mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))$ is

$$
\begin{aligned}
\tau_{1211} &: a_2^1 a_1^1 \mapsto a_1^1 a_2^1 + q^{-2}(1-q^{-2}) a_2^2 a_2^1, \\
\tau_{1121} &: a_1^1 a_1^2 \mapsto a_1^2 a_1^1 - q^{-2}(1-q^{-2}) a_1^2 a_2^2, \\
\tau_{1221} &: a_2^1 a_1^2 \mapsto q^{-2} a_1^2 a_2^1 - q^{-2}(1-q^{-2})(1-(a_2^2)^2), \\
\tau_{2211} &: a_2^2 a_1^1 \mapsto a_1^1 a_2^2, \\
\tau_{1222} &: a_2^1 a_2^2 \mapsto q^{-2} a_2^2 a_2^1, \\
\tau_{2221} &: a_2^2 a_1^2 \mapsto q^{-2} a_1^2 a_2^2, \\
\tau_{1122} &: a_1^1 a_2^2 \mapsto q^{-2} + a_1^2 a_2^1 + (1-q^{-2})(a_2^2)^2.
\end{aligned}
$$

The monomials given in the statement of the result are the reduced monomials with respect to this term rewriting system; furthermore, there are no inclusion ambiguities, and the overlap ambiguities are

$$\begin{aligned}
&(\tau_{1211}, \tau_{1121}, a_2^1, a_1^1, a_1^2), \quad (\tau_{2211}, \tau_{1121}, a_2^2, a_1^1, a_1^2),\\
&(\tau_{1222}, \tau_{2211}, a_2^1, a_2^2, a_1^1), \quad (\tau_{1222}, \tau_{2221}, a_2^1, a_2^2, a_1^2),\\
&(\tau_{2211}, \tau_{1122}, a_2^2, a_1^1, a_2^2), \quad (\tau_{1211}, \tau_{1122}, a_2^1, a_1^1, a_2^2),\\
&(\tau_{1122}, \tau_{2211}, a_1^1, a_2^2, a_1^1), \quad (\tau_{1122}, \tau_{2221}, a_1^1, a_2^2, a_1^2).
\end{aligned}$$

We shall order $\mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))$ with respect to the reduced degree where we give the generators the ordering $a_1^2 < a_1^1 < a_2^2 < a_2^1$. This ordering is compatible with the given term rewriting systems and the rewriting will terminate, so if the ambiguities are resolvable then we can apply the Diamond lemma, and we are done. It can be checked by direct calculation that the ambiguities are resolvable. □

Corollary 3.2.10. *An alternative PBW basis for $A_{\Sigma_{0,4}}$ is*

$$\left\{ (a_1^1)^{\alpha_1}(a_2^1)^{\beta_1}(a_1^2)^{\gamma_1}(a_2^2)^{\delta_1}(b_1^2)^{\alpha_2}(b_1^1)^{\beta_2}(b_2^2)^{\gamma_2}(b_2^1)^{\delta_2}(c_1^1)^{\alpha_3}(c_2^1)^{\beta_3}(c_1^2)^{\gamma_3}(c_2^2)^{\delta_3} \,\middle|\, \alpha_i, \beta_i, \gamma_i \in \mathbb{N}_0, \beta_i \textit{ or } \gamma_i = 0 \right\}.$$

Proof. The same as Proposition 3.2.7 expect we use the PBW basis

$$\left\{ (b_1^2)^{\alpha}(b_1^1)^{\beta}(b_2^2)^{\gamma}(b_2^1)^{\delta} \,\middle|\, \alpha, \beta, \gamma, \delta \in \mathbb{N}_0, \beta \text{ or } \gamma = 0 \right\}$$

from Proposition 3.2.9 for the second copy of $\mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))$ in $A_{\Sigma_{0,4}} = \mathscr{O}(\mathbf{Rep}_q(\mathrm{SL}_2))^{3\otimes}$. □

3.3 The Algebra of Invariants and Character Varieties

Given a surface Σ there are several invariants of Σ based on the representations of the its fundamental group $\pi_1(\Sigma)$.

Definition 3.3.1. The *representation variety* $\mathfrak{R}_G(\Sigma)$ is the affine variety $\mathfrak{R}_G(\Sigma) = \{ \rho : \pi_1(\Sigma) \to G \}$ of homomorphisms from the fundamental group of Σ to the reductive algebraic group G.

Definition 3.3.2. The *character stack* $\underline{\mathrm{Ch}}_G(\Sigma)$ is the quotient $\mathfrak{R}_G(\Sigma)/G$ of the representation variety of the surface $\mathfrak{R}_G(\Sigma)$ by the the group G acting upon it by conjugation.

Definition 3.3.3. The *character variety* $\mathrm{Ch}_G(\Sigma)$ is the affine categorical quotient $\mathfrak{R}_G(\Sigma)//G$ of the representation variety of the surface $\mathfrak{R}_G(\Sigma)$ by the the group G acting upon it by conjugation.

The character stack $\underline{\mathrm{Ch}}_G(\Sigma)$ is intimately related to the factorisation homology of Σ with coefficients in the category $\mathbf{Rep}(G)$ of representations of G:

Theorem 3.3.4. *[BZBJ18a] If Σ is a surface, then we have an equivalence of categories*

$$\mathbf{QCoh}(\underline{\mathrm{Ch}}_G(\Sigma)) \simeq \int_\Sigma \mathbf{Rep}(G)$$

between the category of quasi-coherent sheaves on the character stack $\underline{\mathrm{Ch}}_G(\Sigma)$ *and the factorisation homology of the surface* Σ *with coefficients in* $\mathbf{Rep}(G)$*.*

Proposition 3.3.5. *[BZBJ18a] Let* Σ *be a punctured surface. The algebra object* A_Σ *of* $\int_\Sigma \mathbf{Rep}_q(G)$ *is a quantisation of the character stack* $\underline{\mathrm{Ch}}_G(\Sigma)$*.*

Remark 3.3.6. The character stack $\underline{\mathrm{Ch}}_G(\Sigma)$ is often called the character variety. Another name for the character variety $\mathrm{Ch}_G(\Sigma)$ is the affine character variety.

We now turn our attention to quantising the character variety $\mathrm{Ch}_G(\Sigma)$.

Definition 3.3.7. The *algebra of invariants* $\mathscr{A}_\Sigma$ of the punctured surface Σ with respect to the quantum group $\mathcal{U}_q(()\mathfrak{g})$ is $(\underline{\mathrm{End}}(A_\Sigma))^{\mathcal{U}_q(()\mathfrak{g})}$, the algebra of invariants of A_Σ under the action of $\mathcal{U}_q(()\mathfrak{g})$.

To quantise $\mathrm{Ch}_G(\Sigma)$ we deform the Poisson algebra of functions on $\mathrm{Ch}_G(\Sigma)$, and a suitable deformation of this Poisson algebra is given by $\mathscr{A}_\Sigma$:

Proposition 3.3.8 [BZBJ18a]**.** *Let* Σ *be a punctured surface. The the algebra of invariants* $\mathscr{A}_\Sigma$ *of* $\int_\Sigma \mathbf{Rep}_q(G)$ *is a quantisation of the character variety* $\mathrm{Ch}_G(\Sigma)$*.*

Example 3.3.9. From Section 2.3, we recall that the algebra object $A_{\Sigma_{0,4}}$ is generated by twelve generators

$$\begin{pmatrix} x_1^1 & x_2^1 \\ x_1^2 & x_2^2 \end{pmatrix}$$

for $x \in \{ a, b, c \}$ and where $x_j^i \in V^* \otimes V$. The quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ is generated by $E, F, K^\pm$ whose images in the standard 2–dimensional representation are

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};\ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};\ K = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

It is a Hopf algebra with coproduct Δ defined by

$$\Delta(E) = E \otimes 1 + K^{-1} \otimes E,\ \Delta(F) = F \otimes K + 1 \otimes F,\ \Delta(K) = K \otimes K;$$

antipode S defined by

$$S(E) = KE,\ S(F) = -FK^{-1},\ S(K) = K^{-1};$$

and counit ϵ defined by $\epsilon(E) = \epsilon(F) = 0, \epsilon(K) = 1$. The vector space V with basis $\{ v_1, v_2 \}$ has an $\mathcal{U}_q(\mathfrak{sl}_2)$ action on it defined by

$$\begin{aligned} K \cdot v_1 &= qv_1; & K \cdot v_2 &= q^{-1}v_2; \\ E \cdot v_1 &= 0; & E \cdot v_2 &= v_1; \\ F \cdot v_1 &= v_2; & F \cdot v_2 &= 0. \end{aligned}$$

The action on the dual V^* is defined by $X \cdot u^*(w) = u^*(S(X)w)$ where $X \in \mathcal{U}_q(\mathfrak{sl}_2), u^* \in V^*, w \in V$, so on the basis $\{ v^1, v^2 \}$ is given by

$$K \cdot v^1 = qv^1; \qquad K \cdot v^2 = q^{-1}v^2;$$

$$F \cdot v^1 = -q^{-1}v^2; \quad F \cdot v^2 = 0;$$
$$E \cdot v^1 = 0; \qquad E \cdot v^2 = -qv^1$$

The action of $\mathcal{U}_q(\mathfrak{sl}_2)$ on $V^* \otimes V$ is defined via the coproduct; hence, it acts on $A_{\Sigma_{0,4}}$ as follows:

$$\begin{array}{llll} K \cdot a_1^1 = a_1^1; & K \cdot a_2^1 = q^2 a_2^1; & K \cdot a_1^2 = q^{-2} a_1^2; & K \cdot a_2^2 = a_2^2; \\ E \cdot a_1^1 = q^{-1} a_2^1; & E \cdot a_2^1 = 0; & E \cdot a_1^2 = q(a_2^2 - a_1^1); & E \cdot a_2^2 = -q a_2^1; \\ F \cdot a_1^1 = -q^{-2} a_1^2; & F \cdot a_2^1 = a_1^1 - a_2^2; & F \cdot a_1^2 = 0; & F \cdot a_2^2 = a_1^2. \end{array}$$

An element $x \in A_{\Sigma_{0,4}}$ is an invariant of the $\mathcal{U}_q(\mathfrak{sl}_2)$–action if $h \cdot v = \epsilon(h)v$ i.e. $E \cdot v = F \cdot v = 0$ and $K \cdot v = v$. So, the algebra of invariants quantisation of the SL_2–quantum character variety of $\Sigma_{0,4}$ is given by the elements of $A_{\Sigma_{0,4}}$ which are invariant under this action. We shall give a presentation for $\mathscr{A}_{\Sigma_{0,4}}$ in Section 3.5.

3.4 Hilbert Series Calculations

In this section we shall compute the graded character of the algebra objects $A_{\Sigma_{0,4}}$ and $A_{\Sigma_{1,1}}$, and then use these to compute the Hilbert series of the algebras of invariants $\mathscr{A}_{\Sigma_{0,4}}$ and $\mathscr{A}_{\Sigma_{1,1}}$ which we will need in the proof of presentation of $\mathscr{A}_{\Sigma_{0,4}}$ and $\mathscr{A}_{\Sigma_{1,1}}$ in the next section. A Hilbert series encodes the dimensions of the graded parts of an algebra.

Definition 3.4.1. The *associated graded algebra* of the $\mathbb{Z}_+$ filtered algebra $A = \bigcup_{n \in \mathbb{Z}_+} A(n)$ is

$$\mathscr{G}(A) = \bigoplus_{n \in \mathbb{Z}_+} A[n] \text{ where } A[n] = \begin{cases} A(0) & \text{for } n = 0 \\ A(n) \big/ A(n-1) & \text{for } n > 0. \end{cases}$$

Definition 3.4.2. The *Hilbert series* of the $\mathbb{Z}_+$ graded vector space $A = \bigoplus_{n \in \mathbb{Z}_+} A[n]$ is the formal power series

$$h_A(t) = \sum \dim(A[n])t^n.$$

The Hilbert series of a $\mathbb{Z}_+$ graded algebra A is the Hilbert series of its underlying $\mathbb{Z}_+$ graded vector space, and the Hilbert series of the $\mathbb{Z}_+$ filtered algebra $A = \bigcup_{n \in \mathbb{Z}_+} A(n)$ is the Hilbert series of the associated graded algebra $\mathscr{G}(A)$.

A graded character of a filtered/graded representation encodes the dimensions of graded parts and weight spaces simultaneously.

Definition 3.4.3. Let V be a vector space acted on by $\mathcal{U}_q(\mathfrak{g})$ and let V^k denote the q^k–weight space of V where $k \in \mathbb{Z}$. The *character* of V is the formal power series

$$\mathrm{ch}_V(u) = \sum_{k \in \Lambda} \dim\left(V^k\right) u^k.$$

Definition 3.4.4. Let $V = \bigoplus_n V[n]$ be a graded vector space acted on by $\mathcal{U}_q(\mathfrak{g})$. The *graded character* of V is

$$h_V(u,t) := \sum_n \mathrm{ch}_{V[n]}(u) t^n = \sum_{n,k} \dim\left(V[n]^k\right) u^k t^n,$$

where $V[n]^k$ is the q^k–weight space of $V[n]$. If V is filtered rather than graded the graded character of V $h_V(u,t)$ is $h_{\mathscr{G}(V)}(u,t)$, the graded character of associated graded vector space $\mathscr{G}(V)$.

Let $\Sigma = \Sigma_{0,4}$ or $\Sigma_{1,1}$. Both A_Σ and its subalgebra $\mathscr{A}_\Sigma$ have filtrations by degree:

$$A_\Sigma = \bigcup_{n \in \mathbb{Z}_+} A(n); \ \mathscr{A}_\Sigma = \bigcup_{n \in \mathbb{Z}_+} \mathscr{A}(n)$$

where $A(n)$ and $\mathscr{A}(n)$ are the span of monomials in A_Σ and $\mathscr{A}_\Sigma$ respectively with at most n generators.

Remark 3.4.5. Unless otherwise stated, Hilbert series will always assume grading by degree, and the action of $\mathcal{U}_q(\mathfrak{sl}_2)$ will always be that stated in Example 3.3.9.

As $\mathscr{A}_\Sigma$ is the part of A_Σ with weight $1 = q^0$ under the action of $\mathcal{U}_q(\mathfrak{sl}_2)$, the terms of the graded character $h_{A_\Sigma}(u, v)$ where $k = 0$ give the Hilbert series $h_{\mathscr{A}_\Sigma}(t)$; hence, we shall:

I. Compute the graded character of $\mathscr{O}_q(\mathbf{Rep}_q(\mathrm{SL}_2))$ which we use to

II. Compute the graded character of A_Σ, and then

III. Extract the terms of the graded character which give the Hilbert series of $\mathscr{A}_\Sigma$.

3.4.1 The Graded Character of the Algebra Objects $A_{\Sigma_{0,4}}$ and $A_{\Sigma_{1,1}}$

Proposition 3.4.6. *The graded character of $\mathscr{O}_q(\mathbf{Rep}_q(\mathrm{SL}_2))$ is*

$$h_{\mathscr{O}_q}(u, t) = \frac{(1+t)}{(1-t)(1-u^2t)(1-u^{-2}t)}.$$

Proof. Recall from Proposition 3.2.6 that $\mathscr{O}_q(\mathbf{Rep}_q(\mathrm{SL}_2))$ has basis

$$\left\{ (a_1^1)^\alpha (a_2^1)^\beta (a_1^2)^\gamma (a_2^2)^\delta \;\middle|\; \alpha, \beta, \gamma, \delta \in \mathbb{N}_0;\ \beta \text{ or } \gamma = 0 \right\}.$$

We shall denote $X_{\alpha,\beta,\gamma,\delta} := (a_1^1)^\alpha (a_2^1)^\beta (a_1^2)^\gamma (a_2^2)^\delta$. The n^{th} graded part $\mathscr{O}_q[n] := \left(\mathscr{O}_q(\mathbf{Rep}_q(\mathrm{SL}_2))\right)[n]$ has basis

$$\{ X_{\alpha,\beta,\gamma,\delta} \mid \alpha, \beta, \gamma, \delta \in \mathbb{N}_0;\ \beta \text{ or } \gamma = 0;\ \alpha + \beta + \gamma + \delta = n \}.$$

We can see from Example 3.3.9 that $a_1^1, a_2^1, a_1^2, a_2^2$ have weights $1, q^2, q^{-2}, 1$ respectively, so

$$K \cdot X_{\alpha,\beta,\gamma,\delta} = K \cdot \left((a_1^1)^\alpha (a_2^1)^\beta (a_1^2)^\gamma (a_2^2)^\delta\right) = q^{2\beta - 2\gamma} (a_1^1)^\alpha (a_2^1)^\beta (a_1^2)^\gamma (a_2^2)^\delta = q^{2(\beta-\gamma)} X_{\alpha,\beta,\gamma,\delta},$$

and $X_{\alpha,\beta,\gamma,\delta}$ has weight $q^{2(\beta-\gamma)}$. This means that $\mathscr{O}_q[n]^k$, the q^k weight space of $\mathscr{O}_q[n]$, has basis

$$\{ X_{\alpha,\beta,\gamma,\delta} \mid \alpha, \beta, \gamma, \delta \in \mathbb{N}_0;\ \beta \text{ or } \gamma = 0;\ \alpha + \beta + \gamma + \delta = n;\ 2(\beta - \gamma) = k \}.$$

If k is odd the final condition is never satisfied, and thus $\mathscr{O}_q[n]^k = \emptyset$. If $k = 2m$ for $m \geq 0$ then we get the basis

$$\begin{aligned}
&\{ X_{\alpha,\beta,\gamma,\delta} \mid \alpha, \beta, \gamma, \delta \in \mathbb{N}_0;\ \beta \text{ or } \gamma = 0;\ \alpha + \beta + \gamma + \delta = n;\ 2(\beta - \gamma) = 2m \} \\
&= \{ X_{\alpha,\beta,0,\delta} \mid \alpha, \beta, \gamma, \delta \in \mathbb{N}_0;\ \alpha + \beta + \delta = n;\ \beta = m \} \\
&\text{as } \beta - \gamma \geq 0 \text{ and } \beta \text{ or } \gamma = 0 \text{ implies } \gamma = 0 \\
&= \{ X_{\alpha,m,0,\delta} \mid \alpha, \delta \in \mathbb{N}_0;\ \alpha + \delta = n - m \}.
\end{aligned}$$

which is empty if $m > n$ and has $n-m+1$ elements otherwise. Finally, if $k = -2m$ for $m > 0$ then we get the basis

$$
\begin{aligned}
&\{\, X_{\alpha,\beta,\gamma,\delta} \mid \alpha,\beta,\gamma,\delta \in \mathbb{N}_0;\ \beta \text{ or } \gamma = 0;\ \alpha+\beta+\gamma+\delta = n;\ 2(\beta-\gamma) = -2m \,\} \\
&= \{\, X_{\alpha,0,\gamma,\delta} \mid \alpha,\beta,\gamma,\delta \in \mathbb{N}_0;\ \alpha+\gamma+\delta = n;\ \gamma = m \,\} \\
&\text{as } \beta-\gamma \leq 0 \text{ and } \beta \text{ or } \gamma = 0 \text{ implies } \beta = 0 \\
&= \{\, X_{\alpha,0,m,\delta} \mid \alpha,\delta \in \mathbb{N}_0;\ \alpha+\delta = n-m \,\}.
\end{aligned}
$$

which is empty if $m > n$ and has $n-m+1$ elements otherwise. Hence,

$$
\dim \mathscr{O}_q[n]^k = \begin{cases} n-m+1 & \text{if } k = 2m \text{ for some } m \geq 0 \\ n-m+1 & \text{if } k = -2m \text{ for some } m \geq 1 \\ 0 & \text{otherwise,} \end{cases}
$$

so the character of $\mathscr{O}_q[n]$ is

$$
\begin{aligned}
h_{\mathscr{O}_q[n]}(u) &= \left(\sum_{m=0}^{n} (n-m+1)u^{2m}\right) + \left(\sum_{m=1}^{n} (n-m+1)u^{-2m}\right) \\
&= \frac{u^{-2n}(u^{2+2n}-1)^2}{(u^2-1)^2},
\end{aligned}
$$

and the graded character of $\mathscr{O}_q$ is

$$
h_{\mathscr{O}_q}(u,t) = \sum_{n=0}^{\infty} \frac{u^{-2n}(u^{2+2n}-1)^2}{(u^2-1)^2} t^n = \frac{(1+t)}{(1-t)(1-u^2t)(1-u^{-2}t)}.
$$

□

We note that if $V = \bigoplus_n V(n)$ and $W = \bigoplus_n W(n)$ are two graded vector spaces acted on by $\mathcal{U}_q(\mathfrak{g})$ then $h_{V\otimes W}(u,t) = h_V(u,t) \cdot h_W(u,t)$.

Corollary 3.4.7. *The graded character of $A_{\Sigma_{0,4}}$ is*

$$
h_{A_{\Sigma_{0,4}}}(u,t) = \left(\frac{(1+t)}{(1-t)(1-u^2t)(1-u^{-2}t)}\right)^3.
$$

Proof. We have from Proposition 3.1.19 that $A_{\Sigma_{0,4}} \cong \mathscr{O}_q \otimes \mathscr{O}_q \otimes \mathscr{O}_q$; hence,

$$
h_{A_{\Sigma_{0,4}}}(u,t) = h_{\mathscr{O}_q}(u,t) \cdot h_{\mathscr{O}_q}(u,t) \cdot h_{\mathscr{O}_q}(u,t) = \left(\frac{(1+t)}{(1-t)(1-u^2t)(1-u^{-2}t)}\right)^3.
$$

□

Corollary 3.4.8. *The graded character of $A_{\Sigma_{1,1}}$ is*

$$
h_{A_{\Sigma_{1,1}}}(u,t) = \left(\frac{(1+t)}{(1-t)(1-u^2t)(1-u^{-2}t)}\right)^2.
$$

Proof. We have from Proposition 3.1.19 that $A_{\Sigma_{1,1}} \cong \mathscr{O}_q \otimes \mathscr{O}_q$; hence,

$$h_{A_{\Sigma_{1,1}}}(u,t) = h_{\mathscr{O}_q}(u,t) \cdot h_{\mathscr{O}_q}(u,t) = \left(\frac{(1+t)}{(1-t)(1-u^2t)(1-u^{-2}t)} \right)^2 .$$

$\square$

3.4.2 The Hilbert Series of $\mathscr{A}_{\Sigma_{0,4}}$ and $\mathscr{A}_{\Sigma_{1,1}}$

Proposition 3.4.9. *Let Σ be any punctured surface and A_Σ be the algebra object of $\int_\Sigma \mathbf{Rep}_q(\mathrm{SL}_2)$. The graded character of A_Σ is*

$$h_{A_\Sigma}(u,t) = \sum_{n,k} m_{n,k} \frac{u^{k+1} - u^{k-1}}{u - u^{-1}} t^n$$

for $m_{n,k} \in \mathbb{Z}_+$.

Proof. As integrable representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ are semisimple, any finite–dimensional representation V of $\mathcal{U}_q(\mathfrak{sl}_2)$ when q is generic can be decomposed into $V = \bigoplus_{k \in \mathbb{Z}_+} V[k]^{m_k}$ where $m_k \in \mathbb{Z}_+$ and $V[k]$ is an irreducible representation with character given by the Weyl character formula:

$$\mathrm{ch}_{V(k)} = u^k + u^{k-2} + \cdots + u^{-k+2} + u^{-k} = \frac{u^{k+1} - u^{-k-1}}{u - u^{-1}}.$$

Applying this to $V = A_\Sigma[n]$ the degree n part of $\mathscr{G}(A_\Sigma)$ gives

$$\begin{aligned} h_{A_\Sigma}(u,t) &= h_{\mathscr{G}(A_\Sigma)}(u,t) \\ &= \sum_n \mathrm{ch}_{V[n]}(u) t^n \\ &= \sum_n \mathrm{ch}_{\bigoplus_k V[n](k)^{m_{n,k}}}(u) t^n \\ &= \sum_{n,k} m_{n,k} \, \mathrm{ch}_{V[n](k)}(u) t^n \\ &= \sum_{n,k} m_{n,k} \frac{u^{k+1} - u^{-k-1}}{u - u^{-1}} t^n. \end{aligned}$$

$\square$

Corollary 3.4.10. *Let A_Σ be the algebra object and $\mathscr{A}_\Sigma$ be algebra of invariants of the factorisation homology of $\int_\Sigma \mathbf{Rep}_q(\mathrm{SL}_2)$ for a punctured surface Σ. The Hilbert series $h_{\mathscr{A}_\Sigma}(t)$ is given by the u coefficient of $(u - u^{-1}) \cdot h_{A_\Sigma}(u,t)$.*

Proof. From Proposition 3.4.9 we have that

$$\begin{aligned} h_{A_\Sigma}(u,t) &= \sum_{n,k} m_{n,k} \frac{u^{k+1} - u^{k-1}}{u - u^{-1}} t^n \\ \implies (u - u^{-1}) h_{A_\Sigma}(u,t) &= \sum_{n,k} m_{n,k} (u^{k+1} - u^{k-1}) t^n \end{aligned}$$

where

$$h_{\mathscr{A}_\Sigma}(t) = \sum_n m_{n,0} t^n,$$

so $h_{\mathscr{A}_\Sigma}(t)$ is given by the u coefficient of $(u - u^{-1}) \cdot h_{A_\Sigma}(u, t)$. $\square$

Proposition 3.4.11. *The Hilbert series of $\mathscr{A}_{\Sigma_{0,4}}$ is*

$$h_{\mathscr{A}_{\Sigma_{0,4}}}(t) = \frac{t^2 - t + 1}{(1-t)^6(1+t)^2}.$$

Proof. From Corollary 3.4.7 we have that

$$\begin{aligned}
h_{A_{\Sigma_{0,4}}}(u, t) &= \left(\frac{(1+t)}{(1-t)(1-u^2t)(1-u^{-2}t)}\right)^3 \\
&= \frac{1}{(1-t)^6}\left(\frac{t^3}{(u^2-t)^3} + \frac{3t^2}{(1-t^2)(u^2-t)^2}\right. \\
&\quad + \frac{3(t^2+1)t}{(1-t^2)^2(u^2-t)} + \frac{1}{(1-tu^2)^3} \\
&\quad \left. + \frac{3t^2}{(1-t^2)(1-tu^2)^2} + \frac{3t^2(t^2+1)}{(1-t^2)^2(1-tu^2)}\right)
\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{(1-u^2t)} &= \sum_{i=0}^{\infty}(u^2t)^i = 1 + u^2t + u^4t^2 + \dots \\
\frac{1}{(u^2-t)} &= u^{-2}\sum_{i=0}^{\infty}(u^{-2}t)^i = u^{-2} + u^{-4}t + \dots
\end{aligned}$$

so the u coefficient of $(u - u^{-1}) \cdot h_{A_{\Sigma_{0,4}}}(u, t)$ is

$$\frac{1}{(1-t)^6}\left((1 - 3t) + \frac{3t^2(1-2t)}{(1-t^2)} + \frac{3t^2(1-t)(t^2+1)}{(1-t^2)^2}\right) = \frac{t^2 - t + 1}{(1-t)^6(1+t)^2}$$

which by Corollary 3.4.10 is the Hilbert series of $\mathscr{A}_{\Sigma_{0,4}}$. $\square$

Proposition 3.4.12. *The Hilbert Series of $\mathscr{A}_{1,1}$ is*

$$h_{\mathscr{A}_{\Sigma_{1,1}}} = \frac{1}{(1-t)^3(1+t)}.$$

Proof. From Corollary 3.4.8 we have that

$$\begin{aligned}
h_{A_{\Sigma_{1,1}}}(u, t) &= \left(\frac{(1+t)}{(1-t)(1-u^2t)(1-u^{-2}t)}\right)^2 \\
&= \frac{(1+t)^2}{(1-t)^2(1-t^2)^2}\left(\frac{2t^2}{(1-t^2)(1-tu^2)} + \frac{t^2}{(u^2-t)^2}\right. \\
&\quad \left. + \frac{2t}{(1-t^2)(u^2-t)} + \frac{1}{(1-tu^2)^2}\right),
\end{aligned}$$

so the u coefficient of $(u - u^{-1})h_{A_{\Sigma_{1,1}}}(u, t)$ is

$$\frac{(1+t)^2}{(1-t)^2(1-t^2)^2}\left(\frac{2t^2(1-t)}{(1-t^2)} + (1-2t)\right) = \frac{1}{(1-t)^3(1+t)}$$

which by Corollary 3.4.10 is the Hilbert series of $\mathscr{A}_{\Sigma_{1,1}}$. $\square$

3.5 The Algebra of Invariants of the Four–Punctured Sphere and the Punctured Torus

3.5.1 The Four–Punctured Sphere

We now turn to the first main result of this thesis: giving a presentation of the algebra of invariants $\mathscr{A}_{\Sigma_{0,4}}$ of $\int_{\Sigma_{0,4}} \mathbf{Rep}_q^{\mathrm{fd}}(\mathrm{SL}_2)$. As explained in Section 3.3, this algebra defines a SL_2–quantum character variety of $\Sigma_{0,4}$.

Recall from Section 3.2 that the generators of $A_{\Sigma_{0,4}}$, organised into matrices, are:

$$A := \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix},\ B := \begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix},\ C := \begin{pmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{pmatrix}.$$

Note that the quantum traces $\mathrm{Tr}_q(A) = a_1^1 + q^{-2}a_2^2$, $\mathrm{Tr}_q(B) = b_1^1 + q^{-2}b_2^2$ and $\mathrm{Tr}_q(C) = c_1^1 + q^{-2}c_2^2$ of these matrices are invariant under the action of the quantum group on $\underline{\mathrm{End}}(A_\Sigma)$, and hence are contained in $\mathscr{A}_{\Sigma_{0,4}}$. Furthermore, the quantum trace $tr_q(X)$ of any matrix $X = \sum_i^N A^{\alpha_i}B^{\beta_j}C^{\gamma_i}$ where $\alpha_i, \beta_i, \gamma_i \in \mathbb{N}_0$ is also invariant under the action of the quantum group, so must also be contained in $\mathscr{A}_{\Sigma_{0,4}}$. The quantum Cayley–Hamilton equation $X^2 = \mathrm{Tr}_q(X)X - q^{-2}\det_q(X)$ implies that $tr_q(X)$ is a linear combinations of the traces $\mathrm{Tr}_q(A)$, $\mathrm{Tr}_q(B)$, $\mathrm{Tr}_q(C)$, $\mathrm{Tr}_q(AB)$, $\mathrm{Tr}_q(AC)$, $\mathrm{Tr}_q(BC)$ and $\mathrm{Tr}_q(ABC)$. Therefore, these seven traces generate all the invariants which are of the form $tr_q(X)$. In this section we prove that these seven traces in fact generate the entire algebra of invariants $\mathscr{A}_{\Sigma_{0,4}}$ and state the relations these traces satisfy.

Definition 3.5.1. Let $\mathscr{B}$ be the algebra with generators E, F, G, s, t, u, v subject to the relations:

$$\begin{aligned} FE &= q^2EF + (q^2-q^{-2})G + (1-q^2)(sv+tu), && (3.14)\\ GE &= q^{-2}EG - q^{-2}(q^2-q^{-2})F + (1-q^{-2})(su+tv), && (3.15)\\ GF &= q^2FG + (q^2-q^{-2})E + (1-q^2)(st+uv), && (3.16)\\ EFG &= \begin{cases} -E^2 - q^{-4}F^2 - G^2 - q^{-4}(s^2+t^2+u^2+v^2) \\ +(st+uv)E + q^{-2}(su+tv)F + (sv+tu)G \\ -stuv + q^{-6}(q^2+1)^2 \end{cases} && (3.17) \end{aligned}$$

and s, t, u, v are central.

Theorem 3.5.2. *The map $\Phi' : \mathscr{B} \to \mathscr{A}_{\Sigma_{0,4}}$ defined by:*

$$\begin{aligned} E &\mapsto \mathrm{Tr}_q(AB), & s &\mapsto \mathrm{Tr}_q(A),\\ F &\mapsto \mathrm{Tr}_q(AC), & t &\mapsto \mathrm{Tr}_q(B),\\ G &\mapsto \mathrm{Tr}_q(BC), & u &\mapsto \mathrm{Tr}_q(C),\\ & & v &\mapsto \mathrm{Tr}_q(ABC), \end{aligned}$$

is an isomorphism of algebras. We denote by $\Phi : \mathscr{B} \to \mathscr{O}_q^{3\otimes}$ the map defined by the same formulas.

Before proceeding with the proof of this theorem, we shall find a basis for the algebra $\mathscr{B}$. As the elements u, v, s and t are central, instead of considering $\mathscr{B}$ as an algebra over $\mathbb{C}$ with seven generators, we can consider $\mathscr{B}$ as an algebra over the polynomial ring $\mathbb{C}[s,t,u,v]$with generators E, F, G, i.e. $\mathscr{B} = \mathbb{C}[s,t,u,v]\langle E, F, G\rangle^{\dagger}$.

Proposition 3.5.3. *A PBW–basis for $\mathscr{G}(\mathscr{B})$ over $\mathbb{C}[s,t,u,v]$ is*

$$\left\{ E^n F^m G^l \mid n \text{ or } m \text{ or } l = 0 \right\}.$$

Proof. A term rewriting system for $\mathscr{G}(B)$ is given by

$$\begin{aligned}
\sigma_{FE} &: FE \mapsto q^2 EF + dG + ea \\
\sigma_{GF} &: GF \mapsto q^2 FG + dE + ec \\
\sigma_{GE} &: GE \mapsto q^{-2} EG - q^{-2} dF + fb \\
\sigma_{EF^nG} &: EF^nG \mapsto f(n)
\end{aligned}$$

where

$$a := sv + tu,\ b := su + tv,\ c := st + uv,\ d := (q^2 - q^{-2}),\ e := (1 - q^2),\ f := (1 - q^{-2})$$

and $f(n)$ is defined recursively as follows*:

$$\begin{aligned}
f(1) &:= -E^2 - q^{-4}F^2 - G^2 + cE + q^{-2}bF + aG \\
&\quad + \left(-q^{-4}(s^2 + t^2 + u^2 + v^2) - stuv + q^{-6}(q^2+1)^2\right) \\
f(n) &:= q^{-2} F f(n-1) + (q^{-4} - 1) G F^{n-1} G + (1 - q^{-2}) a F^{n-1} G.
\end{aligned}$$

We shall use the above term rewriting system for $\mathscr{G}(B)$ and apply the Diamond Lemma. In order to do this we must first show that all the ambiguities of the term rewriting system are resolvable. The ambiguities are

1. $(\sigma_{GF}, \sigma_{FE}, G, F, E)$,
2. $(\sigma_{FE}, \sigma_{EF^nG}, F, E, F^nG)$,
3. $(\sigma_{GE}, \sigma_{EF^nG}, G, E, F^nG)$,
4. $(\sigma_{EF^nG}, \sigma_{GE}, EF^n, G, E)$,
5. $(\sigma_{EF^nG}, \sigma_{GF}, EF^n, G, F)$.

The first ambiguity $(\sigma_{GF}, \sigma_{FE}, G, F, E)$ is resolvable by direct calculation:

$$\begin{aligned}
GFE &\xmapsto{\sigma_{GF}} q^2 FGE + dE^2 + ecE \\
&\xmapsto{\sigma_{GE}} FEG - dF^2 + q^2 fbF + dE^2 + ecE
\end{aligned}$$

†The algebra $\langle E, F, G\rangle$ denotes the subalgebra of $\mathscr{B}$ generated by E, F and G not the free algebra.

*This recursion relation arises from applying σ_{FE}^{-1} to EF^nG; one could equally apply σ_{GF}^{-1} which would give an alternate term rewriting system.

$$\xmapsto{\sigma_{FE}} q^2EFG + dG^2 + eaG - dF^2 + q^2fbF + dE^2 + ecE$$

is equal to

$$\begin{aligned}
GFE &\xmapsto{\sigma_{FE}} q^2GEF + dG^2 + eaG \\
&\xmapsto{\sigma_{GE}} EGF - dF^2 + q^2fbF + dG^2 + eaG \\
&\xmapsto{\sigma_{GF}} q^2EFG + dE^2 + ecE - dF^2 + q^2fbF + dG^2 + eaG.
\end{aligned}$$

The second ambiguity $(\sigma_{FE}, \sigma_{EF^nG}, F, E, F^nG)$ also follows directly:

$$\begin{aligned}
FEF^nG &\xmapsto{\sigma_{FE}} q^2EF^{n+1}G + dGF^nG + eaF^nG \\
&\xmapsto{\sigma_{EF^{n+1}G}} Ff(n) - dGF^nG + (q^2-1)aF^nG + dGF^nG + eaF^nG \\
&= Ff(n)
\end{aligned}$$

is equal to

$$FEF^nG \xmapsto{\sigma_{EF^nG}} Ff(n).$$

For the remainder of the ambiguities we proceed by induction on n. For the third ambiguity $(\sigma_{GE}, \sigma_{EF^nG}, G, E, F^nG)$ one direction is given by:

$$\begin{aligned}
GEF^nG &\xmapsto{\sigma_{GE}} q^{-2}EGF^nG - q^{-2}dF^{n+1}G + fbF^nG \\
&\xmapsto{\sigma_{GF}} EFGF^{n-1}G + q^{-2}dE^2F^{n-1}G + q^{-2}ecEF^{n-1}G \\
&\quad - q^{-2}dF^{n+1}G + fbF^nG \\
&\xmapsto{\sigma_{EFG}} \Big(- E^2 - q^{-4}F^2 - G^2 + cE + q^{-2}bF + aG \\
&\quad - q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\Big)F^{n-1}G \\
&\quad + (1-q^{-4})E^2F^{n-1}G + (q^{-2}-1)cEF^{n-1}G - q^{-2}dF^{n+1}G + fbF^nG \\
&= \Big(- q^{-4}E^2 - F^2 - G^2 + q^{-2}cE + bF + aG \\
&\quad - q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\Big)F^{n-1}G \quad \text{for all } n \geq 1 \qquad (\dagger) \\
&\xmapsto{\sigma^2_{EF^{n-1}G}} \Big(- F^2 - G^2 + bF + aG - q^{-4}(s^2+t^2+u^2+v^2) - stuv \\
&\quad + q^{-6}(q^2+1)^2\Big)F^{n-1}G - q^{-4}Ef(n-1) + q^{-2}cf(n-1) \text{ when } n \neq 1. \qquad (\ddagger)
\end{aligned}$$

This equals the other direction when $n = 1$:

$$\begin{aligned}
GEFG &\xmapsto{\sigma_{EFG}} -GE^2 - q^{-4}GF^2 - G^3 + cGE + q^{-2}bGF + aG^2 \\
&\quad - q^{-4}(s^2+t^2+u^2+v^2)G - stuvG + q^{-6}(q^2+1)^2G \\
&\xmapsto{\sigma^3_{GE}} -q^{-4}E^2G + q^{-4}dEF - q^{-2}fbE + q^{-2}dFE - fbE - q^{-4}GF^2 - G^3 \\
&\quad + q^{-2}cEG - q^{-2}dcF + fbc + q^{-2}bGF + aG^2 \\
&\quad + \left(-q^{-4}(s^2+t^2+u^2+v^2) - stuvG + q^{-6}(q^2+1)^2\right) G \\
&\xmapsto{\sigma^3_{GF}} -q^{-4}E^2G + q^{-4}dEF - q^{-2}fbE + q^{-2}dFE - fbE \\
&\quad - F^2G - q^{-2}dFE - q^{-2}ecF - q^{-4}dEF - q^{-4}ecF - G^3
\end{aligned}$$

$$
\begin{aligned}
&+ q^{-2}cEG - q^{-2}dcF + fbc + bFG + q^{-2}dbE + q^{-2}ebc + aG^2\\
&+ \left(-q^{-4}(s^2+t^2+u^2+v^2) - stuvG + q^{-6}(q^2+1)^2\right)G\\
={}& \Big(- q^{-4}E^2 - F^2 - G^2 + q^{-2}cE + bF + aG\\
&- q^{-4}(s^2+t^2+u^2+v^2) - stuvG + q^{-6}(q^2+1)^2\Big)G\\
={}& (\dagger)
\end{aligned}
$$

And in the general case:

$$
\begin{aligned}
GEF^nG &\xmapsto{EF^nG} q^{-2}GFf(n-1) + (q^{-4}-1)G^2F^{n-1}G + (1-q^{-2})aGF^{n-1}G\\
&\xmapsto{\sigma_{GF}} FGf(n-1) + q^{-2}dEf(n-1) + q^{-2}ecf(n-1)\\
&\quad + (q^{-4}-1)G^2F^{n-1}G + (1-q^{-2})aGF^{n-1}G\\
&\mapsto q^{-2}FEGF^{n-1}G - q^{-2}dF^{n+1}G + fbF^nG\\
&\quad + q^{-2}dEf(n-1) + q^{-2}ecf(n-1) + (q^{-4}-1)G^2F^{n-1}G\\
&\quad + (1-q^{-2})aGF^{n-1}G \text{ by the induction assumption}\\
&\xmapsto{\sigma_{FE}} EFGF^{n-1}G + q^{-2}dG^2F^{n-1}G + q^{-2}eaGF^{n-1}G - q^{-2}dF^{n+1}G\\
&\quad + fbF^nG + q^{-2}dEf(n-1) + q^{-2}ecf(n-1) + (q^{-4}-1)G^2F^{n-1}G\\
&\quad + (1-q^{-2})aGF^{n-1}G\\
&\xmapsto{\sigma_{EFG}} \Big(- E^2 - q^{-4}F^2 - G^2 + cE + q^{-2}bF + aG\\
&\quad + \left(-q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\right)\Big)F^{n-1}G\\
&\quad + q^{-2}dG^2F^{n-1}G + q^{-2}eaGF^{n-1}G - q^{-2}dF^{n+1}G + fbF^nG\\
&\quad + q^{-2}dEf(n-1) + q^{-2}ecf(n-1) + (q^{-4}-1)G^2F^{n-1}G\\
&\quad + (1-q^{-2})aGF^{n-1}G\\
&= \Big(- E^2 - F^2 - G^2 + cE + bF + aG + \Big(- q^{-4}(s^2+t^2+u^2+v^2)\\
&\quad - stuv + q^{-6}(q^2+1)^2\Big)\Big)F^{n-1}G + q^{-2}dEf(n-1) + q^{-2}ecf(n-1)\\
&\xmapsto{\sigma^2_{EF^{n-1}G}} \Big(- F^2 - G^2 + bF + aG + \Big(- q^{-4}(s^2+t^2+u^2+v^2)\\
&\quad - stuv + q^{-6}(q^2+1)^2\Big)\Big)F^{n-1}G - q^{-4}Ef(n-1) + q^{-2}cf(n-1)\\
&= (\ddagger)
\end{aligned}
$$

For the forth ambiguity $(\sigma_{EF^nG}, \sigma_{GE}, EF^n, G, E)$, one direction is:

$$
\begin{aligned}
EF^nGE &\xmapsto{\sigma_{GE}} q^{-2}EF^nEG - q^{-2}dEF^{n+1} + fbEF^n\\
&\xmapsto{\sigma_{FE}} EF^{n-1}(EFG + q^{-2}dG^2 + q^{-2}eaG - q^{-2}dF^2 + fbF)\\
&\xmapsto{\sigma_{EFG}} EF^{n-1}\Big(- E^2 - F^2 - q^{-4}G^2 + cE + bF + q^{-2}aG\\
&\qquad - q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\Big).
\end{aligned}
$$

This equals the other direction when $n = 1$:

$$
EFGE \xmapsto{\sigma_{EFG}} -E^3 - q^{-4}F^2E - G^2E + cE^2 + q^{-2}bFE + aGE
$$

$$
\begin{aligned}
&+ \left(-q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\right) E \\
&= E\left(-E^2 + cE - q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\right) E \\
&\quad - q^{-4}F^2E - G^2E + q^{-2}bFE + aGE \\
&\xmapsto{\sigma^3_{FE}\circ\sigma^3_{GE}} E\left(-E^2 + cE - q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\right) \\
&\quad - EF^2 - q^{-2}dGF - q^{-2}eaF - q^{-4}dFG - q^{-4}eaF - q^{-4}EG^2 + q^{-4}dFG \\
&\quad - q^{-2}fbG + q^{-2}dGF - fbG + bEF + q^{-2}dbG + q^{-2}eab + q^{-2}aEG \\
&\quad - q^{-2}daF + fab \\
&= E\Big(- E^2 - F^2 - q^{-4}G^2 + cE + bF + q^{-2}aG - q^{-4}(s^2+t^2+u^2+v^2) \\
&\quad - stuv + q^{-6}(q^2+1)^2\Big).
\end{aligned}
$$

And in the general case:

$$
\begin{aligned}
EF^nGE &\xmapsto{\sigma_{EF^nG}} q^{-2}Ff(n-1)E + (q^{-4}-1)GF^{n-1}GE + (1-q^{-2})aF^{n-1}GE \\
&\mapsto q^{-4}FEF^{n-1}EG - q^{-4}dFEF^n + q^{-2}fbFEF^{n-1} + (q^{-4}-1)GF^{n-1}GE \\
&\quad + (1-q^{-2})aF^{n-1}GE \text{ by the induction assumption} \\
&\xmapsto{\sigma^2_{FE}} EF^{n-1}EFG + q^{-2}dEF^{n-1}G^2 + q^{-2}eaEF^{n-1}G + q^{-4}dGF^{n-1}EG \\
&\quad + q^{-4}eaF^{n-1}EG - q^{-4}dFEF^n + q^{-2}fbFEF^{n-1} + (q^{-4}-1)GF^{n-1}GE \\
&\quad + (1-q^{-2})aF^{n-1}GE \\
&\xmapsto{\sigma_{EFG}} EF^{n-1}\Big(- E^2 - q^{-4}F^2 - q^{-4}G^2 + cE + q^{-2}bF + q^{-2}aG \\
&\quad + \left(-q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\right)\Big) \\
&\quad + q^{-4}dGF^{n-1}EG + q^{-4}eaF^{n-1}EG - q^{-4}dFEF^n + q^{-2}fbFEF^{n-1} \\
&\quad + (q^{-4}-1)GF^{n-1}GE + (1-q^{-2})aF^{n-1}GE \\
&\xmapsto{\sigma^2_{GE}\circ\sigma^2_{FE}} EF^{n-1}\Big(- E^2 - F^2 - q^{-4}G^2 + cE + bF + q^{-2}aG \\
&\quad + \left(-q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\right)\Big).
\end{aligned}
$$

For the final ambiguity $(\sigma_{EF^nG}, \sigma_{GF}, EF^n, G, F)$, one direction is:

$$
\begin{aligned}
EF^nGF &\xmapsto{\sigma_{GF}} q^2EF^{n+1}G + dEF^nE + ecEF^n \\
&\xmapsto{EF^{n+1}G} Ff(n) + q^2(q^{-4}-1)GF^nG + q^2(1-q^{-2})aF^nG + dEF^nE \\
&\quad + ecEF^n.
\end{aligned}
$$

When $n = 1$ this gives

$$
\begin{aligned}
EFGF &\mapsto -FE^2 - q^{-4}F^3 - FG^2 + cFE + q^{-2}bF^2 + aFG \\
&\quad + \left(-q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\right) F \\
&\quad + q^2(q^{-4}-1)GFG + q^2(1-q^{-2})aFG + dEFE + ecEF \\
&\xmapsto{\sigma^3_{FE}} -E^2F - q^{-2}dEG - q^{-2}eaE - dGE - eaE - q^{-4}F^3 - FG^2 + cEF
\end{aligned}
$$

$$
\begin{aligned}
&+ dcG + eac + q^{-2}bF^2 + q^2aFG \\
&+ \left(-q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\right)F + q^2(q^{-4}-1)GFG \\
&\xmapsto{\sigma_{GE}\circ\sigma_{GF}} -E^2F - q^{-2}dEG - q^{-2}eaE - q^{-2}dEG + q^{-2}d^2F - dfb \\
&- eaE - q^{-4}F^3 - FG^2 + cEF + dcG + eac + q^{-2}bF^2 + q^2aFG \\
&+ \left(-q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\right)F - q^2dFG^2 \\
&- d^2EG - decG \\
&= -E^2F + cEF - d^2EG + daE - q^{-4}F^3 + q^{-2}bF^2 - q^4FG^2 + q^2aFG \\
&+ q^{-2}d^2F - q^2dcG - dfb + eac \\
&+ \left(-q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\right)F.
\end{aligned}
$$

This equals the other direction when $n = 1$:

$$
\begin{aligned}
EFGF &\xmapsto{\sigma_{EFG}} -E^2F - q^{-4}F^3 - G^2F + cEF + q^{-2}bF^2 + aGF \\
&+ \left(-q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\right)F \\
&\xmapsto{\sigma_{GE}\circ\sigma_{GF}^3} -E^2F - q^{-4}F^3 - q^4FG^2 - q^2dEG - q^2ecG - q^{-2}dEG \\
&+ q^{-2}d^2F - dfb - ecG + cEF + q^{-2}bF^2 + q^2aFG + daE + eac \\
&+ \left(-q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\right)F \\
&= -E^2F + cEF - d^2EG + daE - q^{-4}F^3 + q^{-2}bF^2 - q^4FG^2 + q^2aFG \\
&+ q^{-2}d^2F - (1+q^2)ecG - dfb + eac \\
&+ \left(-q^{-4}(s^2+t^2+u^2+v^2) - stuv + q^{-6}(q^2+1)^2\right)F.
\end{aligned}
$$

And in the general case:

$$
\begin{aligned}
EF^nGF &\xmapsto{\sigma_{EF^nG}} q^{-2}Ff(n-1)F + (q^{-4}-1)GF^{n-1}GF + (1-q^{-2})aF^{n-1}GF \\
&\mapsto FEF^nG + q^{-2}dFEF^{n-1}E + q^{-2}ecFEF^{n-1} + (q^{-4}-1)GF^{n-1}GF \\
&\quad + (1-q^{-2})aF^{n-1}GF \text{ by the induction assumption} \\
&\xmapsto{\sigma_{GF}^2\circ\sigma_{FE}^2} FEF^nG + dEF^nE + q^{-2}d^2GF^{n-1}E + q^{-2}deaF^{n-1}E \\
&\quad + ecEF^n + q^{-2}decGF^{n-1} + q^{-2}e^2acF^{n-1} + q^2(q^{-4}-1)GF^nG \\
&\quad + (q^{-4}-1)dGF^{n-1}E + (q^{-4}-1)ecGF^{n-1} + q^2(1-q^{-2})aF^nG \\
&\quad + (1-q^{-2})daF^{n-1}E + (1-q^{-2})eacF^{n-1} \\
&= FEF^nG + dEF^nE + ecEF^n - dGF^nG + q^2(1-q^{-2})aF^nG \\
&\xmapsto{\sigma_{EF^nG}} Ff(n) + dEF^nE + ecEF^n - dGF^nG + q^2(1-q^{-2})aF^nG.
\end{aligned}
$$

Hence, all ambiguities in the reduction system are resolvable. It remains to show that the reduction algorithm eventually terminates. We proceed by induction on the degree of the expression. As no rules apply to expressions of degree one, the reduction algorithm trivially terminates. Consider an expression $T \in \mathbb{C}\langle E, F, G\rangle$ of degree n; it is a finite linear combination of words in $\langle E, F, G\rangle$ and can be reduced in a finite number of steps using the reduction rules σ_{FE}, σ_{GE} and σ_{GF} to a finite linear combination of words of the form $E^{\alpha_i}F^{\beta_i}G^{\gamma_i}$ for some $\alpha_i, \beta_i, \gamma_i \in \mathbb{N}_0$ such that $\alpha_i + \beta_i + \gamma_i \leq n$: if each of these monomials is reducible in a finite number of reductions so is T. Either $\beta_i = 0$ and the monomial $E^{\alpha_i}F^{\beta_i}G^{\gamma_i}$ is reduced, or the only reduction we can apply is $\sigma_{EF^{\beta_i}G}$ which reduces the degree, so the result follows by induction. As every expression can be reduced fully in a finite number of reductions and all ambiguities are resolvable, the Diamond Lemma applies giving the result. $\square$

To conclude, a basis of $\mathscr{B}$ over $\mathbb{C}[s,t,u,v]$ is $\left\{ E^nF^mG^l \mid n \text{ or } m \text{ or } l = 0 \right\}$, so a basis of $\mathscr{B}$ over $\mathbb{C}$ is $\left\{ E^nF^mG^ls^at^bu^cv^d \mid n \text{ or } m \text{ or } l = 0; a,b,c,d,n,m,l \in \mathbb{N}_0 \right\}$. We now proceed to the proof of our theorem.

Proof of Theorem 3.5.2. To check that Φ is a morphism of algebras one must check that the images of relations 3.14-3.17 are satisfied in $\mathcal{O}_q^{\otimes 3}$, which is a long but straightforward calculation, which we omit. As all quantum traces lie in $\mathscr{A}_{\Sigma_{0,4}}$, the codomain of Φ can be restricted to define Φ' . So to show Φ' is an isomorphism of algebras it remains to show Φ' is a bijection which will be done by first proving Φ is injective and then proving that Φ' is surjective.

The proof of injectivity of Φ uses a filtration on the codomain $\mathcal{O}_q^{\otimes 3}$.

Definition 3.5.4. We define a filtration on the algebra $\mathcal{O}_q^{\otimes 3} = \bigcup_{i\in\mathbb{N}_0} F_i$ by defining the degree of the generators as follows:

- Degree 0: a_1^2, a_2^2, c_2^1, and c_2^2;
- Degree 1: a_1^1, c_1^1;
- Degree 2: a_2^1,c_1^2, b_1^1, b_2^1, b_1^2, and b_2^2.

Definition 3.5.5. Let $\mathcal{G}(\mathcal{O}_q^{\otimes 3}) = \bigoplus_{n\in\mathbb{N}_0} G_n$ denote the associated graded algebra of $\mathcal{O}_q^{\otimes 3} = \cup_{i\in\mathbb{N}_0} F_i$.

Lemma 3.5.6. *The set*

$$\left\{ \Phi(E^\epsilon F^n G^m s^\alpha t^\beta u^\gamma v^\delta) \mid \epsilon \text{ or } m \text{ or } n = 0; \alpha,\beta,\gamma,\delta,n,m,\epsilon \in \mathbb{N}_0 \right\}$$

is linearly independent in $\mathcal{O}_q^{\otimes 3}$, so the homomorphism $\Phi : \mathscr{B} \to \mathcal{O}_q^{\otimes 3}$ is injective.

Proof. Suppose the contrary that the set $\left\{ \Phi\left(E^\epsilon F^n G^m s^\alpha t^\beta s^\gamma t^\delta\right) \mid \epsilon \text{ or } m \text{ or } n = 0; \epsilon,m,n,\alpha,\beta,\gamma,\delta \in \mathbb{N}_0 \right\}$ is linearly dependent then for some finite indexing set I there exists scalars c_i which are not all zero such that

$$\sum_{i\in I} c_i \Phi(E^{\epsilon_i} F^{n_i} G^{m_i} s^{\alpha_i} t^{\beta_i} u^{\gamma_i} v^{\delta_i}) = 0 \in \mathcal{O}_q^{\otimes 3}. \tag{3.18}$$

Map this to $\mathcal{G}(\mathcal{O}_q^{\otimes 3})$:

$$\sum_{i\in I} c_i \Phi(E^{\epsilon_i} F^{n_i} G^{m_i} s^{\alpha_i} t^{\beta_i} u^{\gamma_i} v^{\delta_i}) = 0 \in \mathcal{G}(\mathcal{O}_q^{\otimes 3}). \tag{3.19}$$

As s,t,u and v are central in $\mathscr{B}$, (3.19) can be rearranged to give

$$\sum_{i\in I} c_i \Phi(s^{\alpha_i} E^{\epsilon_i} v^{\delta_i} t^{\beta_i} F^{n_i} u^{\gamma_i} G^{m_i}) = 0. \tag{3.20}$$

As $\mathcal{G}(\mathcal{O}_q^{\otimes 3})$ is graded, we can assume that all the terms in expression (3.20) are in the maximal degree; we also know that

$$\Phi(X) = \mathrm{Tr}_q(AB) = a_2^1 b_1^2 \qquad\qquad \in \mathcal{G}_4,$$

$$
\begin{aligned}
\Phi(F) &= \mathrm{Tr}_q(AC) = a_2^1 c_1^2 && \in \mathcal{G}_4,\\
\Phi(G) &= \mathrm{Tr}_q(BC) = b_2^1 c_1^2 && \in \mathcal{G}_4,\\
\Phi(s) &= \mathrm{Tr}_q(A) = a_1^1 && \in \mathcal{G}_1,\\
\Phi(t) &= \mathrm{Tr}_q(B) = b_1^1 + q^{-1} b_2^2 && \in \mathcal{G}_2,\\
\Phi(u) &= \mathrm{Tr}_q(C) = c_1^1 && \in \mathcal{G}_1,\\
\Phi(v) &= \mathrm{Tr}_q(ABC) = a_2^1 b_2^2 c_1^2 && \in \mathcal{G}_6,
\end{aligned}
$$

so expression (3.20) implies that:

$$
\sum_{i\in I, S(i)=N} c_i (a_1^1)^{\alpha_i} (a_2^1 b_1^2)^{\epsilon_i} (a_2^1 b_2^2 c_1^2)^{\delta_i} (b_1^1 + b_2^2)^{\beta_i} (a_2^1 c_1^2)^{n_i} (c_1^1)^{\gamma_i} (b_2^1 c_1^2)^{m_i} = 0, \tag{3.21}
$$

where $S(i) := \alpha_i + \gamma_i + 4(\epsilon_i + n_i + m_i + \beta_i) + 6\delta_i$ and $N \in \mathbb{N}_0$. The crossing relations (Corollary 3.2.3):

$$
\begin{array}{llllll}
b_1^1 a_2^1 & = a_2^1 b_1^1 & \in \mathcal{G}_4, & b_1^2 a_2^1 & = q^{-2} a_2^1 b_1^2 & \in \mathcal{G}_4,\\
b_2^2 a_2^1 & = a_2^1 b_2^2 & \in \mathcal{G}_4, & b_2^2 b_1^1 & = b_1^1 b_2^2 & \in \mathcal{G}_4,\\
c_1^1 b_2^1 & = b_2^1 c_1^1 & \in \mathcal{G}_3, & c_2^1 b_2^2 & = b_2^2 c_2^1 & \in \mathcal{G}_2,\\
c_1^2 a_2^1 & = q^{-2} a_2^1 c_1^2 & \in \mathcal{G}_2, & c_1^2 b_1^1 & = b_1^1 c_1^2 & \in \mathcal{G}_4,\\
c_1^2 b_2^1 & = q^{-2} b_2^1 c_1^2 & \in \mathcal{G}_4, & c_1^2 b_2^2 & = b_2^2 c_1^2 & \in \mathcal{G}_4,\\
b_2^2 b_1^1 & = b_1^1 b_2^2 & \in \mathcal{G}_4, & b_2^2 b_2^1 & = q^2 b_2^1 b_2^2 & \in \mathcal{G}_4,\\
c_1^2 c_1^1 & = c_1^1 c_1^2 & \in \mathcal{G}_3, & & &
\end{array}
$$

can be used to reorder the term in expression (3.21) to give

$$
\sum_{\substack{i\in I,\\ S(i)=N}} \sum_{k=0}^{\beta_i} c_i q^{A_{i,k}} (a_1^1)^{\alpha_i} (a_2^1)^{\delta_i+\epsilon_i+\gamma_i} (b_1^2)^{\epsilon_i} (b_1^1)^k (b_2^2)^{\beta_i - k + \delta_i} (b_2^1)^{m_i} (c_1^1)^{\gamma_i} (c_1^2)^{\delta_i + n_i + m_i} = 0, \tag{3.22}
$$

for some constants $A_{i,k} \in \mathbb{Z}$.

Using the basis for $A_{\Sigma_{0,4}}$ given in Lemma 3.2.10, the expression (3.22) is linear combination of distinct monomials which are in the basis of $\mathcal{G}(\mathcal{O}^{\otimes 3})$, so all the coefficients must be zero. This is a contradiction as we assumed that not all the c_i were zero. ☐

In order to prove surjectivity of Φ' we shall give $\mathscr{B}$ a filtration.

Definition 3.5.7. We define a filtration on the algebra $\mathscr{B}$ by defining the degree of the generators as follows:

- Degree 1: s, t, u;
- Degree 2: E, F, G;
- Degree 3: v.

Lemma 3.5.8. *The algebras $\mathscr{B}$ and $\mathscr{A}_{\Sigma_{0,4}}$ have the same Hilbert series when $\mathscr{B}$ is given the filtration defined directly above and $\mathscr{A}_{\Sigma_{0,4}}$ the filtration by degree.*

Proof. The Hilbert series of $\mathscr{A}_{\Sigma_{0,4}}$ was computed in Section 3.4 to be $\frac{1-t+t^2}{(1-t)^6(1+t)^2}$. As

$$
\left\{ E^n F^m G^l s^a t^b u^c v^d \;\middle|\; n \text{ or } m \text{ or } l = 0;\ a,b,c,d,n,m,l \in \mathbb{N}_0 \right\}
$$

is a basis of $\mathscr{G}(\mathscr{B})$, there is a grading preserving vector space isomorphism

$$\begin{aligned} \mathscr{G}(\mathscr{A}) &\to \langle E, F, G\rangle \otimes \mathbb{C}[s] \otimes \mathbb{C}[t] \otimes \mathbb{C}[u] \otimes \mathbb{C}[v] : \\ E^a F^b G^c s^d t^e u^f v^g &\mapsto (E^a F^b G^c) \otimes s^d \otimes t^e \otimes u^f \otimes v^g \end{aligned}$$

where $\langle E, F, G\rangle$ is the subalgebra of $\mathscr{A}$ generated by E, F, G; hence,

$$h_{\mathscr{A}}(t) = h_{\langle E,F,G\rangle}(t) \cdot h_{\mathbb{C}[s]}(t) \cdot h_{\mathbb{C}[t]}(t) \cdot h_{\mathbb{C}[u]}(t) \cdot h_{\mathbb{C}[v]}(t).$$

If $x = s, t, u$ the algebra $\mathbb{C}[x]$ is the polynomial algebra graded by degree, so $(\mathbb{C}[x])[n]$ has basis $\{ x^n \}$, and

$$h_{\mathbb{C}[x]}(t) = \sum_{n=0}^{\infty} (\dim (\mathbb{C}[x])[n]) t^n = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t}.$$

The algebra $\mathbb{C}[v]$ is the polynomial algebra graded by 3 times the degree, so $(\mathbb{C}[x])[n]$ has basis $\left\{ x^{\frac{n}{3}} \right\}$ if $n \equiv 0 \mod 3$ and $\emptyset$ otherwise, and

$$h_{\mathbb{C}[v]}(t) = \sum_{n=0}^{\infty} (\dim (\mathbb{C}[x])[n]) t^n = \sum_{n=0}^{\infty} t^{3n} = \frac{1}{1-t^3}.$$

The algebra $\langle E, F, G\rangle[k]$ has basis

$$\left\{ E^a F^b G^c \;\middle|\; a+b+c = n; a \text{ or } b \text{ or } c \text{ is } 0 \right\}$$

if $k = 2n$ is even and is $\emptyset$ otherwise. Assume k is even so $k = 2n$. If $n = 0$ then the basis has one element $\{ 0 \}$. If $n \neq 0$ then the basis is

$$\begin{aligned} &\left\{ E^a F^b G^c \;\middle|\; a+b+c = n; a \text{ or } b \text{ or } c \text{ is } 0 \right\} \\ &= \left\{ E^a F^b G^c \;\middle|\; a+b+c = n;\ \text{one of } a, b, c \text{ is } 0 \right\} \\ &\quad \sqcup \left\{ E^a F^b G^c \;\middle|\; a+b+c = n;\ \text{two of } a, b, c \text{ is } 0 \right\} \\ &= \left\{ E^a F^b \;\middle|\; a+b = n; a, b \neq 0 \right\} \sqcup \left\{ F^b G^c \;\middle|\; b+c = n; b, c \neq 0 \right\} \\ &\quad \sqcup \left\{ E^a G^c \mid a+c = n; a, c \neq 0 \right\} \sqcup \left\{ E^n, F^n, G^n \right\} \end{aligned}$$

which has $3n$ elements. Hence, the Hilbert series of $\langle E, F, G\rangle$ is

$$h_{\langle E,F,G\rangle}(t) = \sum_{n=0}^{\infty} (\dim (\langle E, F, G\rangle)[n]) t^n = 1 + \sum_{n=1}^{\infty} 3n t^{2n} = 1 + \frac{3t^2}{(1-t^2)^2}.$$

Thus

$$\begin{aligned} h_{\mathscr{A}_{\Sigma_{0,4}}}(t) &= h_{\langle E,F,G\rangle}(t) \cdot h_{\mathbb{C}[s]}(t) \cdot h_{\mathbb{C}[t]}(t) \cdot h_{\mathbb{C}[u]}(t) \cdot h_{\mathbb{C}[v]}(t) \\ &= \left(1 + \frac{3t^2}{(1-t^2)^2}\right) \frac{1}{(1-t)^3(1-t^3)} \\ &= \frac{1-t+t^2}{(1-t)^6(1+t)^2}, \end{aligned}$$

which means that $\mathscr{B}$ and $\mathscr{A}_{\Sigma_{0,4}}$ have the same Hilbert series. $\square$

The homomorphism Φ' is filtered if we give $\mathscr{B}$ the filtration defined in Definition 3.5.7 and

$\mathscr{A}_{\Sigma_{0,4}}$ the filtration by degree. It is injective and the Hilbert series of $\mathscr{B}$ and $\mathscr{A}_{\Sigma_{0,4}}$ are equal, so Φ' is an isomorphism. $\square$

3.5.2 The Punctured Torus

We now obtain a presentation of the algebra of invariants for our second surface, the punctured torus. This is simpler than the four–punctured torus case, and the proofs follow in a similar manner.

Definition 3.5.9. Let $\mathscr{T}$ be the algebra with generators X, Y, Z and relations:

$$\begin{aligned} YX - q^{-1}XY &= (q-q^{-1})Z; \\ XZ - q^{-1}ZX &= -q^{-3}(q-q^{-1})Y; \\ ZY - q^{-1}YZ &= -q^{-3}(q-q^{-1})X. \end{aligned}$$

It has a central element

$$L := q^5 XZY + q^3Y^2 - q^4Z^2 + q^3X^2 - (q-q^{-1}).$$

Proposition 3.5.10. *The monomials*

$$\left\{ X^\alpha Y^\beta Z^\gamma \;\middle|\; \alpha, \beta, \gamma \in \mathbb{N}_0 \right\}$$

are a PBW basis for the algebra $\mathscr{T}$.

Proof. We use the reduced degree with the generators ordered by $X < Y < Z$ as our ordering. From the relations of $\mathscr{T}$ we obtain the term rewriting system

$$\begin{aligned} \sigma_{YX} &: YX \mapsto q^{-1}XY + (q-q^{-1})Z; \\ \sigma_{ZX} &: ZX \mapsto qXZ + q^{-2}(q-q^{-1})Y; \\ \sigma_{ZY} &: ZY \mapsto q^{-1}YZ - q^{-3}(q-q^{-1})X. \end{aligned}$$

this term rewriting system is compatible with the ordering, and its only ambiguity $(\sigma_{ZY}, \sigma_{YX}, Z, X, Y)$ is resolvable, so by the Diamond Lemma the reduced monomials $\left\{ X^\alpha Y^\beta Z^\gamma \;\middle|\; \alpha, \beta, \gamma \in \mathbb{N}_0 \right\}$ form a PBW basis for the algebra. $\square$

Organise the generators of $A_{\Sigma_{1,1}}$ into matrices as follows:

$$A := \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}, B := \begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix}.$$

Theorem 3.5.11. *Define the map $\Psi : \mathscr{T} \to \mathcal{O}_q^{\otimes 2}$ by*

$$\begin{aligned} X &\mapsto \mathrm{Tr}_q(A), \\ Y &\mapsto \mathrm{Tr}_q(B), \\ Z &\mapsto \mathrm{Tr}_q(AB). \end{aligned}$$

The restricted map $\Psi' : \mathscr{T} \to \mathscr{A}_{\Sigma_{1,1}}$ is an algebra isomorphism.

Proof. To check that Ψ is a morphism of algebras one must check that the images of the three relations are satisfied in $\mathcal{O}_q^{\otimes 2}$, which is a long but straightforward calculation. As all quantum traces lie in $\mathscr{A}_{\Sigma_{1,1}}$, the codomain of Ψ can be restricted to define Ψ' . So to show Ψ' is an isomorphism of algebras it remains to show Ψ' is a bijection which will be done by proving Ψ is injective and Ψ' is surjective.

Lemma 3.5.12. *The set*

$$\left\{ \Psi \left(X^\alpha Y^\beta Z^\gamma \right) \,\middle|\, \alpha, \beta, \gamma \in \mathbb{N}_0 \right\}$$

in linearly independent in $\mathcal{O}_q^{\otimes 2}$, so the homomorphism $\Psi : \mathscr{T} \to \mathcal{O}_q^{\otimes 2}$ is injective.

Proof. In this proof we use the filtration in defined in Definition 3.5.4 restricted to $\mathcal{O}_q^{\otimes 2}$. Suppose the contrary to that the set

$$\left\{ \Psi \left(X^\alpha Y^\beta Z^\gamma \right) \,\middle|\, \alpha, \beta, \gamma \in \mathbb{N}_0 \right\}$$

is linearly dependent then for some finite indexing set I there exists scalars c_i which are not all zero such that

$$\sum_{i \in I} c_i \Psi(X^{\alpha_i} Y^{\beta_i} Z^{\gamma_i}) = 0 \in \mathcal{O}_q^{\otimes 2}. \tag{3.23}$$

Map this to $\mathcal{G}(\mathcal{O}_q^{\otimes 2})$:

$$\sum_{i \in I} c_i \Psi(X^{\alpha_i} Y^{\beta_i} Z^{\gamma_i}) = 0 \in \mathscr{G}(\mathcal{O}_q^{\otimes 2}). \tag{3.24}$$

As $\mathcal{G}(\mathcal{O}_q^{\otimes 2})$ is graded, we can assume that all the terms in expression (3.24) are in the maximal degree; we also know that

$$\begin{aligned}
\Phi(X) &= \mathrm{Tr}_q(A) = a^1_1 && \in \mathcal{G}_1, \\
\Phi(Y) &= \mathrm{Tr}_q(B) = b^1_1 + q^{-1} b^2_2 && \in \mathcal{G}_2, \\
\Phi(Z) &= \mathrm{Tr}_q(AB) = a^1_2 b^2_1 && \in \mathcal{G}_4,
\end{aligned}$$

so expression (3.24) implies that:

$$\sum_{i \in I, S(i) = N} c_i (a^1_1)^{\alpha_i} (b^1_1 + q^{-1} b^2_2)^{\beta_i} (a^1_2 b^2_1)^{\gamma_i} = 0, \tag{3.25}$$

where $S(i) := \alpha_i + 4(\beta_i + \gamma_i)$ and $N \in \mathbb{N}_0$. The crossing relations

$$\begin{array}{llllll}
b^1_1 a^1_2 & = a^1_2 b^1_1 & \in \mathcal{G}_4, & b^2_1 a^1_2 & = q^{-2} a^1_2 b^2_1 & \in \mathcal{G}_4, \\
b^2_2 a^1_2 & = a^1_2 b^2_2 & \in \mathcal{G}_4, & b^2_2 b^1_1 & = b^1_1 b^2_2 & \in \mathcal{G}_4, \\
b^2_2 b^1_2 & = q^2 b^1_2 b^2_2 & \in \mathcal{G}_4, & & &
\end{array}$$

can be used to reorder the term in expression (3.25) to give

$$\sum_{\substack{i \in I, \\ S(i) = N}} \sum_{k=0}^{\beta_i} c_i q^{A_{i,k}} (a^1_1)^{\alpha_i} (a^1_2)^{\gamma_i} (b^1_1)^k (b^2_1)^{\gamma_i} (b^2_2)^{\beta_i - k} = 0, \tag{3.26}$$

for some constants $A_{i,k} \in \mathbb{Z}$.

Using the basis for $A_{\Sigma_{1,1}}$ given in Proposition 3.2.8, the expression (3.26) is linear combination of distinct monomials which are in the basis of $\mathcal{G}(\mathcal{O}^{\otimes 2})$, so all the coefficients must be zero. This is a contradiction as we assumed that not all the c_i were zero. $\square$

In order to prove surjectivity of Ψ' we shall give $\mathscr{T}$ a filtration.

Definition 3.5.13. We define a filtration on the algebra $\mathscr{T}$ by defining the degree of the generators as follows:

- Degree 1: X, Y;
- Degree 2: Z.

Lemma 3.5.14. *The associated graded algebra $\mathscr{G}(\mathscr{T})$ has a PBW basis*

$$\left\{ X^\alpha Y^\beta Z^\gamma \;\middle|\; \alpha, \beta, \gamma \in \mathbb{N}_0 \right\}.$$

Proof. The associated graded algebra $\mathscr{G}(\mathscr{T})$ is the algebra with generators X, Y, Z subject to the relations:

$$YX = q^{-1}XY + (q - q^{-1})Z; \quad XZ = q^{-1}ZX; \quad ZY = q^{-1}YZ;$$

We can apply the Diamond Lemma with the above relations as the term rewriting system. $\square$

Lemma 3.5.15. *The algebras $\mathscr{T}$ and $\mathscr{A}_{\Sigma_{1,1}}$ have the same Hilbert series when $\mathscr{T}$ is given the filtration in Definition 3.5.13 and $\mathscr{A}_{\Sigma_{1,1}}$ the filtration by degree.*

Proof. The Hilbert series of $\mathscr{A}_{\Sigma_{1,1}}$ was computed in Section 3.4 to be $\frac{1}{(1-t)^2(1-t^2)}$. We note from Lemma 3.5.14 that

$$\left\{ X^\alpha Y^\beta Z^\gamma \;\middle|\; \alpha, \beta, \gamma \in \mathbb{N}_0 \right\}.$$

is a basis of $\mathscr{G}(\mathscr{T})$, so there is a grading preserving vector space isomorphism

$$\begin{aligned} \mathscr{G}(\mathscr{T}) &\to \mathbb{C}[X] \otimes \mathbb{C}[Y] \otimes \mathbb{C}[X] : \\ X^\alpha Y^\beta Z^\gamma &\mapsto X^\alpha \otimes Y^\beta \otimes Z^\gamma; \end{aligned}$$

hence,

$$h_{\mathscr{T}}(t) = h_{\mathbb{C}[X]}(t) \cdot h_{\mathbb{C}[Y]}(t) \cdot h_{\mathbb{C}[Z]}(t).$$

If $x = X, Y$ the algebra $\mathbb{C}[x]$ is the polynomial algebra graded by degree, so $(\mathbb{C}[x])[n]$ has basis $\{ x^n \}$, and

$$h_{\mathbb{C}[x]}(t) = \sum_{n=0}^{\infty} (\dim(\mathbb{C}[x])[n]) t^n = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t}.$$

The algebra $\mathbb{C}[Z]$ is the polynomial algebra graded by two times the degree, so $(\mathbb{C}[Z])[n]$ has basis $\left\{ Z^{\frac{n}{2}} \right\}$ if $n \equiv 0 \mod 2$ and $\emptyset$ otherwise, and

$$h_{\mathbb{C}[Z]}(t) = \sum_{n=0}^{\infty} (\dim(\mathbb{C}[Z])[n]) t^n = \sum_{n=0}^{\infty} t^{2n} = \frac{1}{1-t^2}.$$

Thus

$$h_{\mathscr{T}}(t) = h_{\mathbb{C}[X]}(t) \cdot h_{\mathbb{C}[Y]}(t) \cdot h_{\mathbb{C}[Z]}(t)$$

$$= \frac{1}{(1-t)^2(1-t^2)},$$

which means that $\mathscr{T}$ and $\mathscr{A}_{\Sigma_{1,1}}$ have the same Hilbert series. □

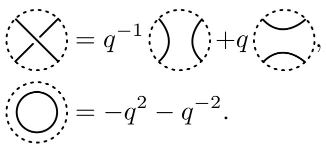
The homomorphism Ψ' is filtered if we give $\mathscr{T}$ the filtration in Lemma 3.5.14 and $\mathscr{A}_{\Sigma_{1,1}}$ the filtration by degree. It is injective and the Hilbert series of $\mathscr{T}$ and $\mathscr{A}_{\Sigma_{1,1}}$ are equal, so Ψ' is an isomorphism. □

3.6 Isomorphisms with Skein Algebras, Spherical Double Affine Hecke Algebras and Cyclic Deformations

In this section we use the presentation of the algebras of invariants $\mathscr{A}_{0,4}$ and $\mathscr{A}_{1,1}$ of the four–punctured sphere $\Sigma_{0,4}$ and punctured torus $\Sigma_{1,1}$ over $\mathcal{U}_q(\mathfrak{sl}_2)$ obtained in the previous section. We state isomorphisms between $\mathscr{A}_{0,4}$ and two isomorphic algebras: $S\mathscr{H}_{q,\underline{t}}$, the spherical double affine Hecke algebra of type $C^\vee C_1$, and $\mathrm{Sk}(\Sigma_{0,4})$, the Kauffman bracket skein algebra of the four–punctured sphere. We also state isomorphisms between $\mathscr{A}_{1,1}$ and two isomorphic algebras: $U_q(\mathfrak{su}_2)$, a cyclic deformation of $U(\mathfrak{su}_2)$, and $\mathrm{Sk}(\Sigma_{1,1})$, the Kauffman bracket skein algebra of the punctured torus.

The Kauffman Bracket Skein Algebra

Definition 3.6.1. The *Kauffman bracket skein module* $\mathrm{Sk}_q(M)$ of an oriented 3–manifold M (possibly with boundary) is the vector space of formal linear sums of isotopy classes of framed links without contractible components in M (but including the empty link) on which we impose the Kauffman bracket skein relations:



Definition 3.6.2. The *Kauffman bracket skein algebra* $\mathrm{Sk}(\Sigma)$ of the surface Σ is the Kauffman bracket skein module $\mathrm{Sk}(\Sigma \times [0,1])$. It is an algebra with multiplication given by stacking copies of $\Sigma \times [0,1]$ on top of each other and retracting.

Theorem 3.6.3. *[BS18, BP00] Let p_i denote the loops around the four punctures of $\Sigma_{0,4}$ and let x_i denote the loops around punctures 1 and 2, 2 and 3, 1 and 3 respectively (see Figure 3.5). The Kauffman bracket skein algebra $\mathrm{Sk}(\Sigma_{0,4})$ has a presentation where the generators are x_i and p_i, and the relations are*

$$[x_i, x_{i+1}]_{q^2} = (q^4 - q^{-4})x_{i+2} - (q^2 - q^{-2})p_i \text{ (indices taken modulo 3)};$$
$$\Omega_K = (q^2 + q^{-2})^2 - (p_1p_2p_3p_4 + p_1^2 + p_2^2 + p_3^2 + p_4^2);$$

where $[a,b]_q := qab - q^{-1}ba$ is the quantum Lie bracket and

$$\Omega_K := -q^2x_1x_2x_3 + q^4x_1^2 + q^{-4}x_2^2 + q^4x_3^2 + q^2p_1x_1 + q^{-2}p_2x_2 + q^2p_3x_3.$$

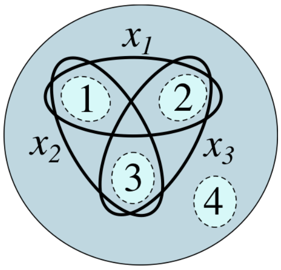


Figure 3.5: The loops x_1, x_2 and x_3

Theorem 3.6.4 [BP00]**.** *The Kauffman bracket Skein algebra* $\mathrm{Sk}(\Sigma_{1,1})$ *has a presentation with generators* x_1, x_2, x_3 *and relations*

$$[x_i, x_{i+1}]_q = (q^2 - q^{-2})x_{i+2} \text{ (indices taken modulo 3).}$$

The Spherical Double Affine Hecke Algebras $S\mathscr{H}_{q,\underline{t}}$ and $SH_{q,t}$, and the Cyclic Deformation of $U(\mathfrak{su}_2)$

Double Affine Hecke Algebras (DAHAs) were introduced by Cherednik [Che92], who used them to prove Macdonald's constant term conjecture for Macdonald polynomials, but have since found wider ranging applications particularly in representation theory [Che04, Che13]. DAHAs can be associated to different root systems with Cherednik's original DAHA being associated to the A^1 root system.

Definition 3.6.5. The A^1 *double affine Hecke algebra* $H_{q,t}$ is the algebra with generators $X^{\pm 1}$, $Y^{\pm 1}$ and T, and relations

$$TXT = X^{-1}, \quad TY^{-1}T = Y, \quad XY = q^2 YXT^2, \quad (T - t)(T + t^{-1}) = 0.$$

The element $e = (T + t^{-1})/(t + t^{-1})$ is an idempotent of $H_{q,t}$, and is used to define the spherical subalgebra $SH_{q,t} := eH_{q,t}e$.

Theorem 3.6.6 [Ter13, Sam14]**.** *The spherical double affine Hecke algebra* $SH_{q,t}$ *has a presentation with generators* x, y, z *and relations*

$$[x, y]_q = (q^2 - q^{-2})z, \quad [z, x]_q = (q^2 - q^{-2})y, \quad [y, z]_q = (q^2 - q^{-2})x$$

$$q^2x^2 + q^{-2}y^2 + q^2z^2 - qxyz = \left(\frac{t}{q} - \frac{q}{t}\right)^2 + \left(q + \frac{1}{q}\right)^2$$

where $[a, b]_q := qab - q^{-1}ba$ *is the quantum Lie bracket.*

The double affine Hecke algebra $\mathscr{H}_{q,\underline{t}}$ of type $C^\vee C_1$ is a 5–parameter deformation of the affine Weyl group $\mathbb{C}[X^\pm, Y^\pm] \rtimes \mathbb{Z}_2$ with deformation parameters $q \in \mathbb{C}^*$ and $\underline{t} = (t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4$. It can be given an abstract presentation with generators are $T_0, T_1, T_0^\vee, T_1^\vee$ and relations:

$$\begin{aligned} (T_0 - t_1)(T_0 + t_1^{-1}) &= 0, \\ (T_0^\vee - t_2)(T_0^\vee + t_2^{-1}) &= 0, \\ (T_1 - t_3)(T_1 + t_3^{-1}) &= 0, \\ (T_1^\vee - t_4)(T_1^\vee + t_4^{-1}) &= 0, \end{aligned}$$

$$T_1^\vee T_1 T_0 T_0^\vee = q.$$

It generalises Cherednik's double affine Hecke algebras of rank 1 as $H_{q;t} := \mathscr{H}_{q,(1,1,t^{-1},1)}$. The element $e = (T_1 + t_3^{-1})/(t_3 + t_3^{-1})$ is an idempotent of $\mathscr{H}_{q,\underline{t}}$, and is used to define the spherical subalgebra $S\mathscr{H}_{q,\underline{t}} := e\mathscr{H}_{q,\underline{t}}e$.

Theorem 3.6.7. *[Ter13, BS18] The spherical double affine Hecke algebra $S\mathscr{H}_{q,\underline{t}}$ has a presentation with generators x, y, z and relations*

$$\begin{aligned}
[x,y]_q &= (q^2 - q^{-2})z - (q - q^{-1})\gamma \\
[y,z]_q &= (q^2 - q^{-2})x - (q - q^{-1})\alpha \\
[z,x]_q &= (q^2 - q^{-2})y - (q - q^{-1})\beta \\
\Omega &= \overline{t_1}^2 + \overline{t_2}^2 + \overline{qt_3}^2 + \overline{t_4}^2 - \overline{t_1}\,\overline{t_2}(\overline{qt_3})\overline{t_4} + (q + q^{-1})^2
\end{aligned}$$

where

$$\begin{aligned}
\alpha &:= \overline{t_1 t_2} + \overline{qt_3 t_4}, \\
\beta &:= \overline{t_1 t_4} + \overline{qt_3 t_2}, \\
\gamma &:= \overline{t_2 t_4} + \overline{qt_3 t_1}, \\
\Omega &:= -qxyz + q^2x^2 + q^{-2}y^2 + q^2z^2 - q\alpha x - q^{-1}\beta y - q\gamma z, \\
[a,b]_q &:= qab - q^{-1}ba \text{ is the quantum Lie bracket.}
\end{aligned}$$

Proposition 3.6.8 [BS18]**.** *There is an isomorphism $\delta : \mathrm{Sk}(\Sigma_{0,4}) \to S\mathscr{H}_{q,\underline{t}}$ given by*

$$\begin{aligned}
\beta(x_1) &= x, & \beta(p_1) &= i\overline{t_1}, \\
\beta(x_2) &= y, & \beta(p_2) &= i\overline{t_2}, \\
\beta(x_3) &= z, & \beta(p_3) &= i\overline{qt_3}, \\
\beta(q) &= q^2, & \beta(p_4) &= i\overline{t_4}.
\end{aligned}$$

Definition 3.6.9 [BP00, Zac90]**.** The *cyclic deformation of* $U(\mathfrak{su}_2)$ is given by

$$U_q(\mathfrak{su}_2) := \mathbb{C}\langle y_1, y_2, y_3 | [y_i, y_{i+1}]_q = y_{i+2} \rangle.$$

where indices are taken modulo 3.

Proposition 3.6.10 [BP00]**.** *When $(q^2 - q^{-2})$ is non–invertible there is an isomorphism*

$$\nu : \mathrm{Sk}(\Sigma_{1,1}) \to U_q(\mathfrak{su}_2) : x_i \mapsto (q^2 - q^{-2})y_i.$$

Note that the element $q^2x_1^2 + q^{-2}x_2^2 + q^2x_3^2 - qx_1x_2x_3$ is central in $U_q(\mathfrak{su}_2)$ and setting it equal to $\left(\frac{t}{q} - \frac{q}{t}\right)^2 + \left(q + \frac{1}{q}\right)^2$ recovers the spherical DAHA $SH_{q,t}$.

Relation to Algebra of Invariants

Proposition 3.6.11. *There is an isomorphism* $\alpha : S\mathscr{H}_{q,\underline{t}} \to \mathscr{A}_{\Sigma_{0,4}}$ *defined by*

$$\begin{aligned}
\alpha(x) &= -qE, & \alpha(\overline{t_1}) &= iqs, \\
\alpha(y) &= -qF, & \alpha(\overline{t_2}) &= iqt, \\
\alpha(z) &= -qG, & \alpha(\overline{qt_3}) &= iqv, \\
& & \alpha(\overline{t_4}) &= iqu.
\end{aligned}$$

Proof. By rewriting the relations in the presentation of $\mathscr{A}_\Sigma$ given in Definition 3.5.1 in terms of the quantum Lie bracket $[\cdot,\cdot]_q$, we see that the algebra of invariants $\mathscr{A}_\Sigma$ has generators E, F, G, u, v, s, t and relations:

$$\begin{aligned}
[E,F]_q &= -q^{-1}(q^2-q^{-2})G + (q-q^{-1})(sv+tu) \\
[F,G]_q &= -q^{-1}(q^2-q^{-2})E + (q-q^{-1})(st+uv) \\
[G,E]_q &= -q^{-1}(q^2-q^{-2})F + (q-q^{-1})(su+tv) \\
\tilde{\Omega} &= -q^2s^2 + -q^2t^2 - q^2u^2 - q^2v^2 - q^4stuv + q^{-2}(q^2+1)^2
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\Omega} = {} & q^4EFG - q^4(st+uv)E - q^2(su+tv)F - q^4(sv+tu)G \\
& + q^4E^2 + F^2 + q^4G^2.
\end{aligned}$$

Also note that

$$\begin{aligned}
\alpha(\Omega) &= \alpha(-qxyz + q^2x^2 + q^{-2}y^2 + q^2z^2 - q\alpha x - q^{-1}\beta y - q\gamma z) \\
&= q^4EFG + q^4E^2 + F^2 + q^4G^2 - q^4(st+uv)E - q^2(su+tv)F - q^4(sv+tu)G \\
&= \tilde{\Omega}.
\end{aligned}$$

The map α is clearly bijective, so it remains to show it is a algebra homomorphism:

$$\begin{aligned}
& \alpha\left([x,y]_q - (q^2-q^{-2})z + (q-q^{-1})\gamma\right) \\
& \quad = q^2[E,F]_q + (q^2-q^{-2})q^2G - (q-q^{-1})q^2(sv+tu) \\
& \quad = q^2\left([E,F]_q + (q^2-q^{-2})G - (q-q^{-1})(sv+tu)\right) \\
& \quad = 0
\end{aligned}$$

and similarly for the next two relations. For the final relation:

$$\begin{aligned}
& \alpha(\overline{t_1}^2 + \overline{t_2}^2 + \overline{qt_3}^2 + \overline{t_4}^2 - \overline{t_1t_2qt_3t_4} + (q+q^{-1})^2) - \Omega)) \\
& \quad = -q^2s^2 - q^2t^2 - q^2v^2 - q^2u^2 - q^4stuv + (q+q^{-1})^2) - \tilde{\Omega} \\
& \quad = 0.
\end{aligned}$$

□

Corollary 3.6.12. *There is an isomorphism* $\beta : \mathrm{Sk}_q(\Sigma_{0,4}) \to \mathscr{A}_{\Sigma_{0,4}}$ *defined by*

$$\beta(x_1) = -qE, \quad \beta(p_1) = -qs,$$

$$\beta(x_2) = -qF, \quad \beta(p_2) = -qt,$$
$$\beta(x_3) = -qG, \quad \beta(p_3) = -qv,$$
$$\beta(q) = q^2, \quad \beta(p_4) = -qu.$$

Proof. Immediate from Proposition 3.6.8. □

Proposition 3.6.13. *There is an isomorphism* $\gamma : \mathscr{A}_{\Sigma_{1,1}} \to \mathrm{Sk}(\Sigma_{1,1})$ *defined by*

$$\gamma(q) = q^2,$$
$$\gamma(X) = iq^{-2}x_2,$$
$$\gamma(Y) = iq^{-2}x_1,$$
$$\gamma(Z) = -q^{-5}x_3.$$

Hence, $\mathscr{A}_{1,1}$ is isomorphic to $U_q(\mathfrak{su}_2)$.

3.7 Isomorphism with a Quantisation of the Moduli Space of Flat Connections

In their paper 'Supersymmetric gauge theories, quantization of $\mathcal{M}_{\text{flat}}$, and conformal field theory', Teschner and Vartanov propose a quantisation for the SL_2–character varieties of surfaces. They state generators and relations for the quantisation of $\mathrm{Ch}_{\mathrm{SL}_2}(\Sigma_{0,4})$ and $\mathrm{Ch}_{\mathrm{SL}_2}(\Sigma_{1,1})$ with the quantisation for other surfaces given by decomposing the surface into such surfaces. In this section we shall briefly outline this decomposition before stating isomorphisms $\mathrm{Ch}_{\mathrm{SL}_2(\mathbb{C})}(\Sigma_{0,4}) \cong \mathscr{A}_{0,4}$ and $\mathrm{Ch}_{\mathrm{SL}_2(\mathbb{C})}(\Sigma_{1,1}) \cong \mathscr{A}_{1,1}$ to the quantisation of the SL_2–character varieties given by algebras of invariants.

Definition 3.7.1. The Poisson algebra of algebraic functions on $\mathrm{Ch}_G(\Sigma)$ is denoted $\mathcal{A}(\Sigma)$.

Definition 3.7.2. We can associate to the Riemann surface Σ a *pants decomposition* $\sigma = (C_\sigma, \Gamma_\sigma)$ where:

1. The *cut system* $C_\sigma = \{\gamma_1, \dots, \gamma_n\}$ is a set of homotopy classes of simple closed curves on Σ such that cutting along these curves produces a pants decomposition

$$\Sigma \backslash C_\sigma \simeq \sqcup_\nu \Sigma_{0,3}^\nu \sqcup_\mu \Sigma_{0,1}^\mu$$

   where the $\Sigma_{0,3}^\nu$ are the 'pairs of pants' and the $\Sigma_{0,1}^\mu$ are discs which are used to fill any unwanted punctures;

2. The *Moore–Seilberg graph* Γ_σ is a 3–valent graph specifying branch cuts, and is needed to distinguish when a Dehn twist has been applied to Σ.

We shall now describe a presentation for $\mathcal{A}(\Sigma)$ which is dependent to a choice of pants decomposition. By Dehn's theorem, a curve γ can be classified uniquely up to homotopy by the *Dehn parameters*

$$\{ (p_i, q_i) \mid i = 1 \dots n \}$$

where p_i and q_i are respectively the intersection number and the twisting number between γ and $\gamma_i \in C_\sigma$.

Each curve $e \in \Gamma_\sigma$ which does not end in the boundary of Σ lies in a subspace Σ_e which is homotopic to either $\Sigma_{0,1}$ or $\Sigma_{1,1}$: if e is a loop then $\Sigma_e \simeq \Sigma_{1,1}$, and if it is not then $\Sigma_e \simeq \Sigma_{0,4}$. To e we assign the curves:

1. $\gamma_s^e := \gamma_e$ is the unique curve $\gamma_e \in C_\sigma$ which lies in the interior of Σ_e; it is the curve in cut system for Σ which also defines a cut system for Σ_e;

2. γ_t^e has Dehn parameters $\{ (p_i^e, 0) \mid i = 1, \dots, n \}$;

3. γ_u^e has Dehn parameters $\{ (p_i^e, \delta_{i,e}) \mid i = 1, \dots, n \}$

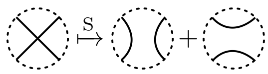
where $p_i^e := \begin{cases} 2\delta_{i,e} & \text{if } \Sigma_e \simeq \Sigma_{0,4} \\ \delta_{i,e} & \text{if } \Sigma_e \simeq \Sigma_{1,1}. \end{cases}$

Definition 3.7.3. Let γ be a closed curve on Σ then its *geodesic length function* is $L_\gamma := \nu_\gamma \operatorname{Tr}_q(\rho(\gamma))$ where ν is a sign and $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2$ is the uniformisation representation.

Remark 3.7.4. The geodesic length functions depend only on the homotopy class of the curve, and the satisfy the 'Skein' relation

$$L_{S(\gamma_1,\gamma_2)} = L_{\gamma_1} L_{\gamma_2}$$

where $S(\gamma_1, \gamma_2)$ is a curve with a crossing point and γ_1, γ_2 are the curves which result from the symmetric smoothing operation:



Proposition 3.7.5. *[VT13] The generators of $\mathcal{A}(\Sigma)$ are*

$$\{ L_s^e, L_t^e, L_u^e \mid e \in \Gamma \textit{ is an interior edge} \}$$

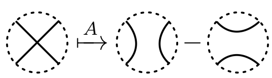
where $L_k^e = |L_{\gamma_k^e}|$. There is a single relation $\mathcal{P}_e(L_s^e, L_t^e, L_u^e)$ on $\mathcal{A}(\Sigma)$ for each internal edge e:

$$\begin{aligned}
\mathcal{P}_e(L_s^e, L_t^e, L_u^e) &= -L_s^e L_t^e L_u^e + (L_s^e)^2 + (L_t^e)^2 + (L_u^e)^2 \\
&\quad + L_s^e(L_3L_4 + L_1L_2) + L_t^e(L_2L_3 + L_1L_4) + L_u^e(L_1L_3 + L_2L_4) \\
&\quad - 4 + L_1^2 + L_2^2 + L_3^2 + L_4^2 + L_1L_2L_3L_4 \textit{ when } \Sigma_e \simeq \Sigma_{0,4}, \textit{ and} \\
\mathcal{P}_e(L_s^e, L_t^e, L_u^e) &= -L_s^e L_t^e L_u^e + (L_s^e)^2 + (L_t^e)^2 + (L_u^e)^2 + L_0 - 2 \textit{ when } \Sigma_e \simeq \Sigma_{1,1},
\end{aligned}$$

where L_1, L_2, L_2, L_4 are loops around the four punctures of $\Sigma_{0,4}$, and L_0 is a loop around the single puncture of $\Sigma_{1,1}$. The Poisson bracket on $\mathcal{A}(\Sigma)$ is given by

$$\{ L_{\gamma_1}, L_{\gamma_2} \} = L_{A(\gamma_1,\gamma_2)},$$

where A is the antisymmetric smoothing operation:



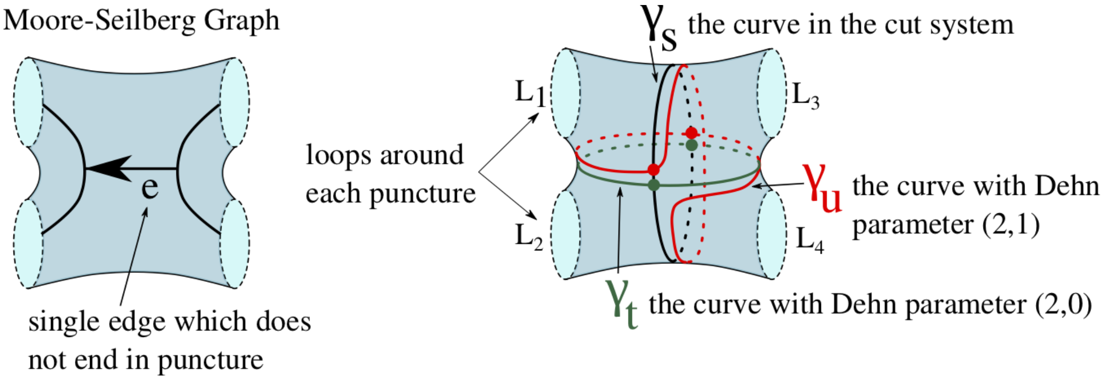


Figure 3.6: Applied to the four–punctured sphere.

As $\mathcal{A}(\Sigma)$ is given by local data on copies of $\Sigma_{0,4}$ and $\Sigma_{1,1}$, Teschner and Vartanov state the deformation for these basic surfaces.

Proposition 3.7.6 [VT13]. *The deformation $\mathcal{A}_b(\Sigma_{0,4})$ of $\mathscr{A}(\Sigma_{0,4})$ is generated by $L_s, L_t, L_u, L_1, L_2, L_3, L_4$ with relations*

$$\begin{aligned}
\mathcal{Q}_e(L_s, L_t, L_u) &= e^{\pi i b^2} L_s L_t - e^{-\pi i b^2} L_t L_s \\
&\quad - (e^{2\pi i b^2} - e^{-2\pi i b^2}) L_u - (e^{\pi i b^2} - e^{-\pi i b^2})(L_1 L_3 + L_2 L_4) \\
\mathcal{P}_e(L_s, L_t, L_u) &= -e^{\pi i b^2} L_s L_t L_u + e^{2\pi i b^2} L_u^2 + e^{2\pi i b^2} L_s^2 + e^{-2\pi i b^2} L_t^2 \\
&\quad + e^{\pi i b^2}(L_1 L_3 + L_2 L_4) L_u + e^{\pi i b^2}(L_3 L_4 + L_2 L_1) L_s \\
&\quad + e^{-\pi i b^2}(L_1 L_4 + L_2 L_3) L_t + L_1^2 + L_3^2 + L_2^2 + L_4^2 + L_1 L_3 L_2 L_4 \\
&\quad - (2\cos(\pi b^2))^2
\end{aligned}$$

where the quadratic relations $\mathcal{Q}_e$ arise from deforming the Poisson bracket.

Proposition 3.7.7 [VT13]. *The deformation $\mathcal{A}_b(\Sigma_{1,1})$ of $\mathscr{A}(\Sigma_{1,1})$ is generated by L_s, L_t, L_u, L_0 with relations*

$$\begin{aligned}
\mathcal{Q}_e(L_s, L_t, L_u) &= e^{\frac{\pi i}{2}} L_s L_t - e^{-\frac{\pi i}{2}} L_t L_s - (e^{\pi i b^2} - e^{-\pi i b^2}) L_u \\
\mathcal{P}_e(L_s, L_t, L_u) &= e^{\pi i b^2} L_s^2 + e^{-\pi i b^2} L_t^2 + e^{\pi i b^2} L_u^2 - e^{\frac{\pi i}{2}} L_s L_t L_u + L_0 - 2\cos(\pi b^2)
\end{aligned}$$

Using the presentation for the algebras of invariants $\mathscr{A}_{\Sigma_{0,4}}$ and $\mathscr{A}_{\Sigma_{1,1}}$ from Section 3.5, we see that we have the following isomorphisms:

Proposition 3.7.8. *The algebra of invariants $\mathscr{A}_{\Sigma_{0,4}}$ is isomorphic to $\mathcal{A}_b(\Sigma_{0,4})$ with isomorphism $\iota : \mathscr{A}_{\Sigma_{0,4}} \to \mathcal{A}_b(\Sigma_{0,4})$ defined by*

$$\begin{aligned}
\iota(q) &= e^{i\pi b^2}, & \iota(s) &= e^{-i\pi b^2} L_1, \\
\iota(E) &= -e^{-i\pi b^2} L_u, & \iota(t) &= e^{-i\pi b^2} L_3, \\
\iota(F) &= -e^{-i\pi b^2} L_s, & \iota(v) &= e^{-i\pi b^2} L_2, \\
\iota(G) &= -e^{-i\pi b^2} L_t, & \iota(u) &= e^{-i\pi b^2} L_4.
\end{aligned}$$

Proof. The map $\kappa : S\mathscr{H}_{q,\underline{t}} \to \mathcal{A}_b(\Sigma_{0,4})$ defined by

$$q \mapsto e^{i\pi b^2}, \qquad \overline{t_1} \mapsto iL_1,$$

$$
\begin{aligned}
x &\mapsto L_u, & \overline{t_2} &\mapsto iL_3,\\
y &\mapsto L_s, & \overline{qt_3} &\mapsto iL_2,\\
z &\mapsto L_t, & \overline{t_4} &\mapsto iL_4,
\end{aligned}
$$

maps $S\mathscr{H}_{q,\underline{t}}$ to an algebra generated by L_s, L_t, L_u with relations

$$
\begin{aligned}
0 &= e^{\pi ib^2}L_uL_s - e^{-\pi ib^2}L_sL_u - (e^{2\pi ib^2} - e^{-2\pi ib^2})L_t - (e^{\pi ib^2} - e^{-\pi ib^2})(L_1L_4 + L_2L_3)\\
0 &= e^{\pi ib^2}L_sL_t - e^{-\pi ib^2}L_tL_s - (e^{2\pi ib^2} - e^{-2\pi ib^2})L_u - (e^{\pi ib^2} - e^{-\pi ib^2})(L_1L_3 + L_2L_4)\\
0 &= e^{\pi ib^2}L_tL_u - e^{-\pi ib^2}L_uL_t - (e^{2\pi ib^2} - e^{-2\pi ib^2})L_s - (e^{\pi ib^2} - e^{-\pi ib^2})(L_3L_4 + L_2L_1)\\
0 &= -e^{\pi ib^2}L_sL_tL_u + e^{2\pi ib^2}L_u^2 + e^{2\pi ib^2}L_s^2 + e^{-2\pi ib^2}L_t^2\\
&\quad + e^{\pi ib^2}(L_1L_3 + L_2L_4)L_u + e^{\pi ib^2}(L_3L_4 + L_2L_1)L_s + e^{-\pi ib^2}(L_1L_4 + L_2L_3)L_t\\
&\quad + L_1^2 + L_3^2 + L_2^2 + L_4^2 + L_1L_3L_2L_4 - (2\cos(\pi b^2))^2
\end{aligned}
$$

which is just the algebra $\mathcal{A}_b(\Sigma_{0,4})$. Hence the algebra $\mathscr{A}_{\Sigma_{0,4}}$ is isomorphic to both $S\mathscr{H}_{q,\underline{t}}$ and $\mathcal{A}_b(\Sigma_{0,4})$ and isomorphism $\iota : \mathscr{A}_{\Sigma_{0,4}} \to \mathcal{A}_b(\Sigma_{0,4})$ is given by $\kappa \circ \alpha^{-1}$. $\square$

Proposition 3.7.9. *The algebra of invariants $\mathscr{A}_{\Sigma_{1,1}}$ is isomorphic to $\mathcal{A}_b(\Sigma_{1,1})$ with isomorphism $\mu : \mathscr{A}_{\Sigma_{1,1}} \to \mathcal{A}_b(\Sigma_{1,1})$ defined by*

$$
\begin{aligned}
\mu(Y) &= iq^{-1}s\\
\mu(X) &= iq^{-1}t\\
\mu(Z) &= -q^{-\frac{5}{2}}u\\
\mu(L) &= L_0
\end{aligned}
$$

Chapter 4

Relative Tensor Products, Skein Categories and Factorisation Homology

The goal of this chapter is to prove that skein categories satisfy excision, and hence to show that they are k–linear factorisation homologies whose free cocompletions recover the presentable factorisation homologies considered in the previous chapter.

We begin by proving that the colimit of the 2–sided bar construction in $\mathbf{Cat}_k$ is the relative tensor product of k–linear categories which was defined by Tambara [Tam01] and has a concrete description. This colimit defines the relative tensor product in k–linear factorisation homology used in the statement of excision, so to prove excision of skein categories it suffices to prove excision where the relative tensor product is the relative tensor product of k–linear categories. Then in Section 4.2 we define skein categories and prove that skein categories satisfy excision. Finally in Section 4.3 we use the results of the previous two sections to conclude that skein categories are k–linear factorisation homologies and relate them to presentable factorisation homologies.

4.1 Relative Tensor Products

4.1.1 Relative Tensor Product of k-linear Categories

The definition of the relative tensor product of k–linear categories is a categorical analogue of the definition of the relative tensor product of modules. The definition of the relative tensor product of modules can be reformulated as follows:

Definition 4.1.1. Let R be a ring, M be a right R–module, N be a left R–module, and G be an abelian group.

1. A homomorphism $f : M \times N \to G$ is R*–balanced* if it is linear and $f(m \cdot r, n) = f(m, r \cdot n)$ for all $r \in R, m \in M, n \in N$.

2. The abelian group $\mathrm{Bal}_R(M, N; G)$ is the set of all R–balanced homomorphisms $M \times N \to G$ with the sum and inverses of balanced homomorphisms defined pointwise, i.e.

$(-f)(m,n) := -f(m,n)$ and $(f+g)(m,n) := f(m,n)+g(m,n)$ for all $f, g \in \mathrm{Bal}_R(M, N; G)$, $m \in M, n \in N$.

3. The *relative tensor product* $M \otimes_R N$ is an abelian group satisfying the universal property that $\mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N, G) \cong \mathrm{Bal}_R(M, N; G)$ for all abelian groups G.

Instead of being relative to a ring A, a relative tensor product $\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}$ of k–linear categories is relative to a monoidal k–linear category $\mathscr{A}$. Instead of being A–modules, $\mathscr{M}$ and $\mathscr{N}$ must be $\mathscr{A}$–module categories.

Definition 4.1.2. Let $\mathscr{A}$ be a monoidal k–linear category. A *left $\mathscr{A}$–module category* is a k–linear category $\mathscr{M}$ equipped with a k–bilinear functor

$$\triangleright : \mathscr{A} \otimes \mathscr{M} \to \mathscr{M} : (a, m) \mapsto a \triangleright m,$$

a natural isomorphism

$$\beta : \_ \triangleright (\_ \triangleright \_) \to (\_ \otimes \_) \triangleright \_ \text{ with components } \beta_{a,b,m} : a \triangleright (b \triangleright m) \to (a \otimes b) \triangleright m$$

called the *associator*, and a natural isomorphism

$$\eta : 1_{\mathscr{A}} \triangleright \_ \to \_ \text{ with components } \eta_m : 1_{\mathscr{A}} \triangleright m \to m$$

called the *unitor* which make the following diagrams commute for all $a, b, c \in \mathscr{A}$ and $m \in \mathscr{M}$

$$\begin{tikzcd}
 & a \triangleright (b \triangleright (c \triangleright m)) \arrow[dl, "\beta_{a,b,c\triangleright m}"'] \arrow[dr, "\mathrm{Id}_a \triangleright \beta_{b,c,m}"] & \\
(a \otimes b) \triangleright (c \triangleright m) \arrow[d, "\beta_{a\otimes b,c,m}"'] & & a \triangleright ((b \otimes c) \triangleright m) \arrow[d, "\beta_{a,b\otimes c,m}"] \\
((a \otimes b) \otimes c) \triangleright m \arrow[rr, "\alpha_{a,b,c} \triangleright \mathrm{Id}_m"] & & (a \otimes (b \otimes c)) \triangleright m
\end{tikzcd}
\qquad
\begin{tikzcd}
a \triangleright (1_{\mathscr{A}} \triangleright m) \arrow[r, "\beta_{a,1_{\mathscr{A}},m}"] \arrow[d, "\mathrm{Id}_a \triangleright \eta_m"'] & (a \otimes 1_{\mathscr{A}}) \triangleright m \arrow[dl, "\rho_a \triangleright m"] \\
a \triangleright m &
\end{tikzcd}$$

The definition for a right $\mathscr{A}$–module category is analogous.

Tambara in [Tam01] defines the relative tensor product $\mathscr{M} \boxtimes_{\mathscr{C}} \mathscr{M}$ of the right $\mathscr{C}$–module category $\mathscr{M}$ and the left $\mathscr{C}$–module category $\mathscr{N}$ relative to the monoidal category $\mathscr{A}$.

Definition 4.1.3. A bilinear functor $F : \mathscr{M} \times \mathscr{N} \to \mathscr{C}$ is *$\mathscr{A}$–balanced* if there is a natural isomorphism which on components is

$$\iota_{m,a,n} : F(m \triangleleft a, n) \to F(m, a \triangleright n)$$

satisfying the commutative diagram

$$\begin{tikzcd}[column sep=small]
& ((m \lhd a) \lhd b, n) \arrow[dl, "\iota_{m \lhd a, b, n}"'] \arrow[dr, "(\beta_{m,a,b}, \mathrm{Id}_n)"] & \\
(m \lhd a, b \rhd n) \arrow[d, "\iota_{m,a,b \rhd n}"'] & & (m \lhd (a * b), n) \arrow[d, "\iota_{m, a*b, n}"] \\
(m, a \rhd (b \rhd n)) \arrow[rr, "(\mathrm{Id}_m, \beta_{a,b,n})"] & & (m, (a * b) \rhd n)
\end{tikzcd}$$

for all $m \in \mathscr{M}$, $a, b \in \mathscr{A}$ and $n \in \mathscr{N}$.

Definition 4.1.4. The natural transformation $\alpha : F \Rightarrow G$ of $\mathscr{A}$–balanced functors $F, G : \mathscr{M} \otimes \mathscr{N} \to \mathscr{C}$ is a *$\mathscr{A}$–balanced natural transformation* if it is compatible with the balancings, i.e the following diagram commutes

$$\begin{tikzcd}
F(m \lhd a, n) \arrow[r, "(\iota_F)_{m,a,n}"] \arrow[d, "\alpha_{(m \lhd a, n)}"] & F(m, a \rhd n) \arrow[d, "\alpha_{(m, a \rhd n)}"] \\
G(m \lhd a, n) \arrow[r, "(\iota_G)_{m,a,n}"] & G(m, a \rhd n)
\end{tikzcd}$$

Definition 4.1.5. We denote the category of $\mathscr{A}$–balanced functions $\mathscr{M} \times \mathscr{N} \to \mathscr{C}$ with $\mathscr{A}$–balanced natural transformations are morphisms by $\mathbf{Fun}_{\mathscr{A}\text{–bal}}(\mathscr{M}, \mathscr{N}; \mathscr{C})$.

Definition 4.1.6 [Tam01]**.** Let $\mathscr{C}$ be a k–linear monoidal category, let $\mathscr{M}$ be a right $\mathscr{C}$–module k–linear category, and let $\mathscr{N}$ be a left $\mathscr{C}$–module k–linear category. The *relative tensor product* of $\mathscr{M}$ and $\mathscr{N}$ relative to $\mathscr{A}$ is a k–linear category $\mathscr{M} \boxtimes_{\mathscr{A}} \mathscr{N}$ together with a $\mathscr{A}$–balanced functor $P : \mathscr{M} \times \mathscr{N} \to \mathscr{C}$ such that for all k–linear categories $\mathscr{C}$ there is an equivalence of categories

$$\mathbf{Fun}_{\mathscr{A}\text{–bal}}(\mathscr{M}, \mathscr{N}; \mathscr{C}) \simeq \mathbf{Cat}_k(\mathscr{M} \boxtimes_{\mathscr{A}} \mathscr{N}, \mathscr{C})$$

given by composing functors in with P.

Tambara then shows the existence of such a relative tensor product by constructing it.

Definition 4.1.7 [Tam01]**.** Let $\mathscr{A}$ be a k–linear monoidal category, let $\mathscr{M}$ be a k–linear right $\mathscr{A}$–module category, and let $\mathscr{N}$ be a k–linear left $\mathscr{A}$–module category. The *relative tensor product* $\mathscr{M} \boxtimes_{\mathscr{A}} \mathscr{N}$ is the k–linear category with the following generators and relations. The objects of $\mathscr{M} \boxtimes_{\mathscr{A}} \mathscr{N}$ are tuples (m, n) where $m \in \mathscr{M}$ and $n \in \mathscr{N}$. The morphisms of $\mathscr{M} \boxtimes_{\mathscr{A}} \mathscr{N}$ are generated by morphisms (f, g), where $f : m \to m'$ is a morphism in $\mathscr{M}$ and $g : g \to g'$ is a morphism in $\mathscr{N}$, and by morphisms $\iota_{m,a,n} : (m \lhd a, n) \to (m, a \rhd n)$ and $\iota^{-1}_{m,a,n} : (m, a \rhd n) \to (m \lhd a, n)$, where $m \in \mathscr{M}$, $a \in \mathscr{A}$ and $n \in \mathscr{N}$. The morphisms satisfy the following relations:

Linearity $(f + f', g) = (f, g) + (f', g)$, $(f, g + g') = (f, g) + (f, g')$ and $a(f, g) = (af, g) = (f, ag)$;

Functionality $(f'f, g'g) = (f', g') \circ (f, g)$ and $(\mathrm{Id}_m, \mathrm{Id}_n) = \mathrm{Id}_{(m,n)}$;

Isomorphism $\iota_{m,a,n} \circ \iota^{-1}_{m,a,n} = \mathrm{Id}_{(m, a \rhd n)}$ and $\iota^{-1}_{m,a,n} \circ \iota_{m,a,n} = \mathrm{Id}_{(m \lhd a, n)}$;

Naturality $\iota_{m',a',n'} \circ (f \lhd u, g) = (f, u \rhd g) \circ \iota_{m,a,n}$

Pentagon and Triangle

$$\begin{tikzcd}[column sep=small]
& ((m \lhd a) \lhd b, n) \arrow[dl, "\iota_{m \lhd a, b, n}"'] \arrow[dr, "(\beta_{m,a,b}, \mathrm{Id}_n)"] & \\
(m \lhd a, b \rhd n) \arrow[d, "\iota_{m,a,b \rhd n}"'] & & (m \lhd (a * b), n) \arrow[d, "\iota_{m, a*b, n}"] \\
(m, a \rhd (b \rhd n)) \arrow[rr, "(\mathrm{Id}_m, \beta_{a,b,n})"] & & (m, (a * b) \rhd n)
\end{tikzcd}
\qquad
\begin{tikzcd}
(m \lhd \mathrm{Id}_{\mathscr{A}}, n) \arrow[r, "\iota_{m, \mathrm{Id}_{\mathscr{A}}, n}"] \arrow[d, "(\theta_m, \mathrm{Id}_n)"'] & (m, \mathrm{Id}_{\mathscr{A}} \rhd n) \arrow[dl, "(\mathrm{Id}_m, \eta_n)"] \\
(m, n) &
\end{tikzcd}$$

The $\mathscr{A}$–balanced bilinear functor $P : \mathscr{M} \times \mathscr{N} \to \mathscr{M} \boxtimes_{\mathscr{A}} \mathscr{N}$ is defined by $P(m,n) = (m,n)$ on objects and $P(f,g) = (f,g)$ on morphisms.

4.1.2 Bicolimits

All of the 2–categories we consider in this thesis are strict; however, we shall use weak colimits (bicolimits) rather than strict 2–colimits. These definitions can be found in [Str72, Lei04, B6́7].

Definition 4.1.8. Let $F, G : \mathscr{C} \to \mathscr{D}$ be 2–functors between 2–categories. A *lax natural transformation* $\sigma : F \to G$ consists of:

1. for each object x in $\mathscr{C}$, a morphism $\sigma_x : F(x) \to G(x)$;

2. for each pair of objects (x,y) in $\mathscr{C}$, a natural transformation

$$\sigma_{x,y} : (\sigma_x)^* \circ G(x,y) \to (\alpha_y)_* \circ F(x,y)$$

where $(\sigma_x)^*$ and $(\alpha_y)_*$ are defined by pre and post composing with σ:

$$(\sigma_y)_* : \begin{tikzcd} F(x) \arrow[r, bend left=50, "F(f)"{name=U}] \arrow[r, bend right=50, "F(g)"'{name=D}] & F(y) \arrow[from=U, to=D, Rightarrow, "\xi"] \end{tikzcd} \mapsto \begin{tikzcd} F(x) \arrow[r, bend left=50, "F(f)"{name=U}] \arrow[r, bend right=50, "F(g)"'{name=D}] & F(y) \arrow[r, "\sigma_y"] \arrow[from=U, to=D, Rightarrow, "\xi"] & G(y) \end{tikzcd}$$

$$(\sigma_x)^* : \begin{tikzcd} G(x) \arrow[r, bend left=50, "G(f)"{name=U}] \arrow[r, bend right=50, "G(g)"'{name=D}] & G(y) \arrow[from=U, to=D, Rightarrow, "\xi"] \end{tikzcd} \mapsto \begin{tikzcd} F(x) \arrow[r, "\sigma_x"] & G(x) \arrow[r, bend left=50, "G(f)"{name=U}] \arrow[r, bend right=50, "G(g)"'{name=D}] & G(y) \arrow[from=U, to=D, Rightarrow, "\xi"] \end{tikzcd}$$

such that,

1. or every object x of $\mathscr{C}$, σ_{1_x} is the identity natural transformation, and

2. for every pair of morphisms $(f,g) \in \mathscr{C}(y,z) \times \mathscr{C}(x,y)$, $\sigma_{f \circ g} = (\sigma_f F(g)) \circ (G(f)\sigma_g)$.

A *pseudonatural transformation* is a lax natural transformation whose 2–cells are invertible, so in $(2,1)$–categories pseudonatural transformations and lax natural transformations are the same.

Remark 4.1.9. Lax and pseudonatural transformation are usually defined to be between pseudofunctors (functors between bicategories whose compatiblity morphisms are invertible). In this case the conditions σ must satisfy are more complex as there are associators and unitors to take into consideration.

Definition 4.1.10. Given two lax–natural transformations

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] & \mathscr{D} \end{tikzcd} \quad \zeta \Downarrow \; \Downarrow \eta$$

a *modification* $\Gamma : \zeta \to \eta$ assigns to each object $x \in \mathscr{C}$ a 2–morphism

$$\begin{tikzcd} F(x) \arrow[r, bend left=50, "\zeta(x)", ""{name=U, below}] \arrow[r, bend right=50, "\eta(x)"', ""{name=D, above}] & G(x) \arrow[Rightarrow, from=U, to=D, "\Gamma(x)"] \end{tikzcd}$$

in $\mathscr{D}$ such that for all morphisms $f : x \to y$ in $\mathscr{C}$ the following diagram commutes

$$\begin{tikzcd} G(f) \circ \zeta(x) \arrow[r, "G(f)\Gamma_x"] \arrow[d, "\zeta(f)"] & G(f) \circ \eta(x) \arrow[d, "\eta(f)"] \\ \zeta(y) \circ F(f) \arrow[r, "\Gamma(f)F(f)"] & \eta(y) \circ F(f) \end{tikzcd}$$

Remark 4.1.11. The definition of a modification between 2–natural transformations is identical to the definition for lax natural transformations, so one can extend 2**Cat**, the 2–category of 2–categories, to a (strict) 3–category by taking modifications as 3–morphisms.

Definition 4.1.12. Let $\mathscr{C}$ be a $(2,1)$–category and let $\mathscr{D}$ be a small $(2,1)$–category. A *diagram* X of shape $\mathscr{D}$ in $\mathscr{C}$ is a lax functor $X : \mathscr{D} \to \mathscr{C}$.

Remark 4.1.13. We shall assume that $X : \mathscr{D} \to \mathscr{C}$ is always a strict 2–functor.

Definition 4.1.14. Let $\mathbf{Diag}_{\mathscr{D}}(\mathscr{C})$ denote the $(2,1)$–category of diagrams of shape J in $\mathscr{C}$:

1. The objects are diagrams of shape J in $\mathscr{C}$;
2. The 1–morphisms are pseudonatural transformations;
3. The 2–morphisms are modifications.

Definition 4.1.15. Let x be an object of $\mathscr{C}$ and let $\mathscr{X}$ denote the $(2,1)$–category with an single object x, a single 1–morphism 1_x and a single 2–morphism which is the trivial 2–cell $1 : 1_y \to 1_y$. The *trace functor* on c is the 2–functor $\mathrm{Tr}_q(x) : \mathscr{C} \to \mathscr{X}$ which sends all objects in $\mathscr{C}$ to y, all 1–morphisms to 1_x, and all 2–morphisms to 1.

Definition 4.1.16. Let $\mathscr{C}$ be a $(2,1)$–category. Denote by $\mathrm{Tr}_q : \mathscr{C} \to \mathbf{Diag}_{\mathscr{D}}(\mathscr{C})$ the 2–functor which sends an object $x \in \mathscr{C}$ to the trace functor $\mathrm{Tr}_q(x)$, a 1–morphism $f : x \to y$ to the trivial pseudonatural transformation $\Gamma : \mathrm{Tr}_q(x) \to \mathrm{Tr}_q(y)$, and a 2–morphism to the trivial modification $\sigma : \Gamma \to \Gamma$.

Definition 4.1.17. Let $\mathscr{C}$ be a 2–category and let X be a diagram of shape J in $\mathscr{C}$. The 2–*colimit of* X is an object $\mathrm{Bicolim}(X)$ in $\mathscr{C}$ together with a pseudonatural equivalence between $\mathrm{Hom}_{\mathscr{C}}(\mathrm{Bicolim}(X), \_) : \mathscr{C} \to \mathscr{C}$ and $\mathbf{Diag}_{\mathscr{D}}(\mathscr{C})(X, \mathrm{Tr}_q(\_)) : \mathscr{C} \to \mathscr{C}$.

Definition 4.1.18. If $\mathscr{C}$ is a $(2,1)$–category then a 2–colimit of $F : X \to \mathscr{C}$ is called a *(2,1)–colimit.*

4.1.3 Colimits of the Truncated Bar Construction

In the next section we give relative tensor product of k-linear categories as the bicolimit in $\mathbf{Cat}_k$ of the truncated bar construction. Before doing this we briefly expand the definition of a bicolimit of the shape of the truncated bar construction.

Definition 4.1.19. Let $\mathscr{D}$ be the 2–category

$$\overline{A} \overset{\overline{g}_1}{\underset{\overline{g}_3}{\overset{\overline{g}_2}{\rightrightarrows\!\!\!\!\to}}} \overline{B} \underset{\overline{f}_2}{\overset{\overline{f}_1}{\rightrightarrows}} \overline{C}$$

with 2–cells

$$\overline{\kappa}_1 : \overline{f}_2 \circ \overline{g}_1 \to \overline{f}_1 \circ \overline{g}_3, \quad \overline{\kappa}_2 : \overline{f}_1 \circ \overline{g}_1 \to \overline{f}_1 \circ \overline{g}_2, \quad \overline{\kappa}_3 : \overline{f}_2 \circ \overline{g}_3 \to \overline{f}_2 \circ \overline{g}_2;$$

and let $X : \mathscr{D} \to \mathscr{C}$ be a (strict) 2–functor to a 2–category $\mathscr{C}$. The image of an object, 1–morphisms or 2–morphisms under X is denoted without a bar, so

$$X\left(\overline{A} \overset{\overline{g}_1}{\underset{\overline{g}_3}{\overset{\overline{g}_2}{\rightrightarrows\!\!\!\!\to}}} \overline{B} \underset{\overline{f}_2}{\overset{\overline{f}_1}{\rightrightarrows}} \overline{C} \right) = A \overset{g_1}{\underset{g_3}{\overset{g_2}{\rightrightarrows\!\!\!\!\to}}} B \underset{f_2}{\overset{f_1}{\rightrightarrows}} C \quad \text{and } X(\overline{\kappa}_i) = \kappa_i.$$

Recall from Section 4.1.2 that the colimit of X is an object $\mathrm{Bicolim}(X)$ in $\mathscr{C}$ together with a pseudonatural equivalence $\Gamma : \mathrm{Hom}_{\mathscr{C}}(\mathrm{Bicolim}(X), \_) \to \mathbf{Diag}_{\mathscr{D}}(\mathscr{C})(X, \mathrm{Tr}_q(\_))$. This means that for all $Y \in \mathscr{C}$ there is an equivalence of categories

$$\Gamma_Y : \mathrm{Hom}_{\mathscr{C}}(\mathrm{Bicolim}(X), Y) \to \mathbf{Diag}_{\mathscr{D}}(\mathscr{C})(X, \mathrm{Tr}_q(Y)),$$

so in order to understand $\mathrm{Bicolim}(X)$ we shall first look at $\mathbf{Diag}_{\mathscr{D}}(\mathscr{C})(X, \mathrm{Tr}_q(Y))$.

Proposition 4.1.20. *The category $\mathbf{Diag}_{\mathscr{D}}(\mathscr{C})(X, \mathrm{Tr}_q(Y))$ has objects of the form*

$$\sigma = \begin{pmatrix} \sigma_A : A \to Y \\ \sigma_B : B \to Y \\ \sigma_C : C \to Y \\ \sigma_{f_i} : \sigma_B \to \sigma_C \circ f_i \\ \sigma_{g_j} : \sigma_A \to \sigma_B \circ g_j \end{pmatrix}$$

where $i = 1, 2$ and $j = 1, 2, 3$, which satisfy the relations:

$$\sigma_C \kappa_1 = \Delta_{13} \Delta_{21}^{-1},$$
$$\sigma_C \kappa_2 = \Delta_{12} \Delta_{11}^{-1},$$
$$\sigma_C \kappa_3 = \Delta_{22} \Delta_{23}^{-1}$$

where $\Delta_{ij} := (\sigma_{f_i} g_j)\sigma_{g_j}$. *The morphisms of* $\mathbf{Diag}_{\mathscr{D}}(\mathscr{C})(X, \mathrm{Tr}_q(Y))$ *are natural isomorphisms*

$$\begin{tikzcd} C \arrow[d, bend right, "\sigma_C"'] \arrow[d, bend left, "\eta_C"] \arrow[d, phantom, "\xRightarrow{\Gamma}"] \\ Y, \end{tikzcd}$$

satisfying the relations

$$\begin{aligned} \eta_{f_1}^{-1}(\Gamma f_1)\sigma_{f_1} &= \eta_{f_2}^{-1}(\Gamma f_2)\sigma_{f_2} \\ (\Delta^{\eta}_{ij})^{-1}(\Gamma f_i g_j)\Delta^{\sigma}_{ij} &= (\Delta^{\eta}_{kl})^{-1}(\Gamma f_k g_l)\Delta^{\sigma}_{kl} \end{aligned}$$

where $i, k = 1, 2$ *and* $j, l = 1, 2, 3$.

Proof. This proof amounts to unravelling the definitions. An object of $\mathbf{Diag}_{\mathscr{D}}(\mathscr{C})(X, \mathrm{Tr}_q(Y))$ is a pseudonatural transformation $\sigma : X \to \mathrm{Tr}_q(Y)$. By the definition of a pseudonatural transformation we have

1. for every $\overline{X} \in \mathscr{D}$, that is $\overline{X} = \overline{A}, \overline{B}, \overline{C}$, 1–morphisms $\sigma_{\overline{X}} : S(\overline{X}) \to Tr(Y)(\overline{X})$: we shall usually denote these morphisms simply as σ_X as they are morphisms $\sigma_X : X \to Y$ for $X = A, B, C$;

2. for every pair of objects $(\overline{W}, \overline{X})$ in $\mathscr{D}$, a natural transformation

$$\sigma_{\overline{W},\overline{X}} : (\sigma_W)^* \circ \mathrm{Tr}_q(Y)(\overline{W}, \overline{X}) \to (\sigma_{\overline{X}})_* \circ S(\overline{W}, \overline{X}).$$

   This means that for every 1–morphism $\overline{h} : \overline{W} \to \overline{X}$ in $\mathscr{D}$, that is $\overline{h} = \overline{g}_1, \overline{g}_2, \overline{g}_3, \overline{f}_1, \overline{f}_2, \overline{f_i} \circ \overline{g_j}, 1_W$, there is a 2–morphism

$$\sigma_h : \sigma_W \to \sigma_X \circ h.$$

   As we are working with $(2,1)$–categories, σ_h is automatically a 2–isomorphism. As $\sigma_{\overline{W},\overline{X}}$ is natural, we have for every 2–morphism $\overline{\kappa} : \overline{h} \to \overline{l}$, that is $\overline{\kappa} = \overline{\kappa}_1, \overline{\kappa}_2, \overline{\kappa}_3, \mathrm{Id}_{\overline{h}}$, that the following diagram commutes

$$\begin{tikzcd} \sigma_W \arrow[d, "\sigma_h"] \arrow[dr, "\sigma_l"] & \\ \sigma_X \circ h \arrow[r, "\sigma_Y \circ \kappa"] & \sigma_X \circ l \end{tikzcd}$$

   This result is trivial for $\mathrm{Id}_{\overline{h}}$, so let $\overline{\kappa} = \overline{\kappa}_1, \overline{\kappa}_2$ or, $\overline{\kappa}_3$. In which case, $\overline{W} = \overline{A}$, $\overline{X} = \overline{C}$, $l = f_i \circ g_j$ and $h = f_k \circ g_l$. As σ_h is invertible, we have that

$$\sigma_C \circ \kappa = \sigma_{f_i \circ g_j} \sigma^{-1}_{f_k \circ g_l}$$

3. for every object $\overline{X}$ of $\mathscr{D}$, $\sigma_{1_{\overline{X}}}$ is the identity natural isomorphism

4. for every composition of morphisms $f \circ g$ in $\mathscr{D}$, $\sigma_{f \circ g} = (\sigma_f g)(\sigma_g)$

So we have that $\sigma : X \to \mathrm{Tr}_q(y)$ consists of 1–morphisms $\sigma_X : X \to Y$ for $X = A, B, C$; and 2–morphisms $\sigma_h : \sigma_W \to \sigma_X \circ h$ for $h = f_i, g_j : W \to X$ such that $\pi_D \kappa_1 = \Delta_{13}\Delta_{21}^{-1}$, $\pi_D \kappa_2 = \Delta_{12}\Delta_{11}^{-1}$ and $\pi_D \kappa_3 = \Delta_{22}\Delta_{23}^{-1}$ where $\Delta_{ij} = (\sigma_{f_i} g_j)\sigma_{g_j}$.

A morphism $\Gamma : \sigma \to \eta$ in $\mathbf{Diag}_{\mathscr{D}}(\mathscr{C})(X, \mathrm{Tr}_q(Y))$ is a modification between σ and η. The modification Γ assigns to each object $\overline{X} \in \mathscr{D}$, that is $\overline{X} = \overline{A}, \overline{B}, \overline{C}$, a 2–morphism

$$\begin{tikzcd} X \arrow[r, bend left, "\sigma_X"] \arrow[r, bend right, "\eta_X"'] \arrow[r, phantom, "\Downarrow\Gamma_X"] & Y \end{tikzcd}$$

such that the following diagram commutes for all $\overline{h} : \overline{W} \to \overline{X}$

$$\begin{tikzcd} \sigma_W \arrow[r, "\Gamma_W"] \arrow[d, "\sigma_h"] & \eta_W \arrow[d, "\eta_h"] \\ \sigma_X \circ h \arrow[r, "\Gamma_X h"] & \eta_X \circ h \end{tikzcd}$$

As all 2–cells are invertible, applying this relation to the f_is gives

$$\Gamma_B = \eta_{f_i}^{-1}(\Gamma_C f_i)\sigma_{f_i}$$

and then applying this relation to the g_is gives

$$\begin{aligned}
\Gamma_A &= \eta_{g_j}^{-1}(\Gamma_B g_j)\sigma_{g_j} \\
&= \eta_{g_j}^{-1}\Big(\big(\eta_{f_i}^{-1}(\Gamma_C f_i)\sigma_{f_i}\big)g_j\Big)\sigma_{g_j} \text{ substituting } \Gamma_B = \eta_{f_i}^{-1}(\Gamma_C f_i)\sigma_{f_i} \\
&= \eta_{g_j}^{-1}(\eta_{f_i}^{-1}g_j)(\Gamma_C f_i g_j)(\sigma_{f_i}g_j)\sigma_{g_j} \text{ as } \big(\eta_{f_i}^{-1}(\Gamma_C f_i)\sigma_{f_i}\big)g_j = (\eta_{f_i}^{-1}g_j)(\Gamma_C f_i g_j)(\sigma_{f_i}g_j) \\
&= (\Delta_{ij}^{\eta})^{-1}(\Gamma_C f_i g_j)\Delta_{ij}^{\sigma}
\end{aligned}$$

from which we conclude that it is sufficient to define Γ_C and that the relation for g_j is automatically satisfied if it is for the compositions $f_i \circ g_j$. □

Remark 4.1.21. The morphisms σ_A, σ_B and σ_C fit into the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "g_1"] \arrow[r, "g_2"] \arrow[r, shift right=2, "g_3"'] \arrow[dr, bend right, "\sigma_A"'] & B \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[d, "\sigma_B"] & C \arrow[dl, bend left, "\sigma_C"] \\ & Y & \end{tikzcd},$$

and the natural isomorphisms σ_{f_i} and σ_{g_j} are 2-cells in this diagram.

4.1.4 The Relative Tensor Product as a Colimit

Definition 4.1.22. Let $\mathscr{A}$ be a monoidal k–linear category and let $\mathscr{M}, \mathscr{N}$ be left/right $\mathscr{A}$–module k–linear categories. The *truncated bar construction* is the diagram

$$\begin{tikzcd} \mathscr{M} \otimes \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{N} \arrow[r, shift left=2, "G_1"] \arrow[r, "G_2"] \arrow[r, shift right=2, "G_3"'] & \mathscr{M} \otimes \mathscr{A} \otimes \mathscr{N} \arrow[r, shift left, "F_1"] \arrow[r, shift right, "F_2"'] & \mathscr{M} \otimes \mathscr{N} \end{tikzcd}$$

where

$$\begin{aligned}
G_1 : \mathscr{M} \otimes \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{N} : &\quad G_1(m, a, b, n) = (m \triangleleft a, b, n); \\
G_2 : \mathscr{M} \otimes \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{N} : &\quad G_2(m, a, b, n) = (m, a * b, n); \\
G_3 : \mathscr{M} \otimes \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{N} : &\quad G_3(m, a, b, n) = (m, a, b \triangleright n);
\end{aligned}$$

$$F_1 : \mathscr{M} \otimes \mathscr{A} \otimes \mathscr{N} : \qquad F_1(m,a,n) = (m \triangleleft a, n);$$
$$F_2 : \mathscr{M} \otimes \mathscr{A} \otimes \mathscr{N} : \qquad F_2(m,a,n) = (m, a \triangleright n);$$

with m, a, b, n objects or morphisms in the categories $\mathscr{M}, \mathscr{A}, \mathscr{A}, \mathscr{N}$ respectively, and there are two cells

$$\kappa_1 : F_2 \circ G_1 \to F_1 \circ G_3$$
$$\kappa_2 : F_1 \circ G_1 \to F_1 \circ G_2$$
$$\kappa_3 : F_2 \circ G_3 \to F_2 \circ G_2$$

where κ_1 is the identity and $\kappa_2(n,a,b,m) : ((n \triangleleft a) \triangleleft b, m) \to (n \triangleleft (a * b), m)$ and $\kappa_3(n,a,b,m) : (n, a \triangleright (b \triangleright m)) \to (n, (a * b) \triangleright m)$ are given by the associators of the $\mathscr{A}$ action.

Theorem 4.1.23. *The relative tensor product $\mathscr{M} \boxtimes_{\mathscr{A}} \mathscr{N}$ of the right $\mathscr{C}$–module k–linear category $\mathscr{M}$ and the left $\mathscr{C}$–module k–linear category $\mathscr{N}$ relative to the k-linear monoidal category $\mathscr{A}$ is the bicolimit of the diagram*

$$\mathscr{M} \otimes \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{N} \overset{\xrightarrow{G_1}}{\underset{\xrightarrow{G_3}}{\xrightarrow{G_2}}} \mathscr{M} \otimes \mathscr{A} \otimes \mathscr{N} \overset{\xrightarrow{F_1}}{\underset{F_2}{\longrightarrow}} \mathscr{M} \otimes \mathscr{N}$$

with 2-cells κ_1, κ_2, κ_3 defined above.

Proof. By the definition of a bicolimit there is an equivalence of categories

$$\Gamma_{\mathscr{C}} : \mathbf{Cat}_c(\mathrm{Bicolim}(X), \mathscr{C}) \to \mathbf{Diag}_{\mathscr{D}}(\mathbf{Cat}_k)(X, \mathrm{Tr}_q(\mathscr{C})),$$

so if there is an equivalence of categories

$$I_{\mathscr{C}} : \mathbf{Diag}_{\mathscr{D}}(\mathbf{Cat}_k)(X, \mathrm{Tr}_q(\mathscr{C})) \to \mathbf{Fun}_{\mathscr{A}\text{–bal}}(\mathscr{M}, \mathscr{N}; \mathscr{C})$$

for every $\mathscr{C} \in \mathbf{Cat}_k$ then by Definition 4.1.7 $\mathrm{Bicolim}(X)$ is the relative tensor product $\mathscr{M} \boxtimes_{\mathscr{A}} \mathscr{N}$.

We shall now define $I_{\mathscr{C}}$ and show it to be a equivalence of categories. Let σ be an object of $\mathbf{Diag}_{\mathscr{D}}(\mathbf{Cat}_k)(X, \mathrm{Tr}_q(\mathscr{C}))$, so

$$\sigma = \begin{pmatrix} \sigma_A : A \to Y \\ \sigma_B : B \to Y \\ \sigma_C : C \to Y \\ \sigma_{F_i} : \sigma_B \to \sigma_C \circ F_i \\ \sigma_{G_j} : \sigma_A \to \sigma_B \circ G_j \end{pmatrix}$$

where $A := \mathscr{M} \times \mathscr{A} \times \mathscr{A} \times \mathscr{N}$, $B := \mathscr{M} \times \mathscr{A} \times \mathscr{N}$ and $C := \mathscr{M} \times \mathscr{N}$. We define

$$I_{\mathscr{C}}(\sigma) := \sigma_C \text{ with balancing } \alpha : \sigma_C F_i \Rightarrow \sigma_C F_2 : \quad \alpha := \sigma_{F_2} \sigma_{F_1}^{-1}.$$

A morphism of $\mathbf{Diag}_{\mathscr{D}}(\mathbf{Cat}_k)(X, \mathrm{Tr}_q(\mathscr{C}))$ is a natural isomorphism $\Gamma : \sigma \to \eta$, we define

$$I_{\mathscr{C}}(\Gamma) := \Gamma.$$

$I_{\mathscr{C}}$ is a well-defined functor

This requires one to prove two things: α is an $\mathscr{A}$–balancing and Γ is a natural transformation of $\mathscr{A}$–balanced functors. To show α is a balancing of σ_C we must show that the diagram

$$\begin{tikzcd}
& ((m \lhd a) \lhd b, n) \arrow[dl, "\alpha_{m \lhd a,b,n} = (\alpha G_1)(m,a,b,n)"'] \arrow[dr, "\sigma_C(\kappa_2)_{m,a,b,n}"] & \\
(m \lhd a, b \rhd n) \arrow[d, "\alpha_{m,a,b \rhd n} = (\alpha G_3)(m,a,b,n)"'] & & (m \lhd (a * b), n) \arrow[d, "\alpha_{m,a*b,n} = (\alpha G_2)(m,a,b,n)"] \\
(m, a \rhd (b \rhd n)) \arrow[rr, "\sigma_C(\kappa_3)_{m,a,b,n}"] & & (m, (a * b) \rhd n)
\end{tikzcd}$$

commutes for all $(m, a, b, n) \in \mathscr{M} \times \mathscr{A} \times \mathscr{A} \times \mathscr{N}$. This is the case as

$$\begin{aligned}
&(\sigma_C\kappa_3)(\alpha G_3)(\sigma_C\kappa_1)(\alpha G_1)(\sigma_C\kappa_2^{-1}) \\
&= \Delta_{22}\Delta_{23}^{-1}(\sigma_{F_2}G_3)(\sigma_{F_1}^{-1}G_3)\Delta_{13}\Delta_{21}^{-1}(\sigma_{F_2}G_1)(\sigma_{F_1}^{-1}G_1)\Delta_{11}\Delta_{12}^{-1} \\
&\quad \text{by definition of } \alpha \text{ and compatibility relations of } \sigma \\
&= (\sigma_{F_2}G_2)\sigma_{G_2}\sigma_{G_3}^{-1}(\sigma_{F_2}^{-1}G_3)(\sigma_{F_2}G_3)(\sigma_{F_1}^{-1}G_3)(\sigma_{F_1}G_3)\sigma_{G_3}\sigma_{G_1}^{-1} \\
&\quad (\sigma_{F_2}^{-1}G_1)(\sigma_{F_2}G_1)(\sigma_{F_1}^{-1}G_1)(\sigma_{F_1}G_1)\sigma_{G_1}\sigma_{G_2}^{-1}(\sigma_{F_1}^{-1}G_2) \\
&\quad \text{by definition of } \Delta_{ij} \\
&= (\sigma_{F_2}G_2)(\sigma_{F_1}^{-1}G_2) \\
&\quad \text{cancelling terms} \\
&= (\sigma_{F_2}\sigma_{F_1}^{-1})G_2 \\
&= \alpha G_2
\end{aligned}$$

Hence, σ_C is $\mathscr{A}$–balanced with balancing α.

Now we shall show that the natural transformation $\Gamma_C : \sigma_C \to \eta_C$ is a natural transformation of $\mathscr{A}$–balanced functors. To show this me must show that that following diagram commutes:

$$\begin{tikzcd}
\sigma_C(m \lhd a, n) \arrow[r, "(\alpha_\sigma)_{m,a,n}"] \arrow[d, "(\Gamma_C)_{(m \lhd a,n)}"] & \sigma_C(m, a \rhd n) \arrow[d, "(\Gamma_C)_{(m,a \rhd n)}"] \\
\eta_C(m \lhd a, n) \arrow[r, "(\alpha_\eta)_{m,a,n}"] & \eta_C(m, a \rhd n)
\end{tikzcd}$$

This is the case as by the definition of Γ we have that

$$\begin{aligned}
&\eta_{F_1}^{-1}(\Gamma_C F_1)\sigma_{F_1} = \eta_{F_2}^{-1}(\Gamma_C F_2)\sigma_{F_2} \\
\implies &\eta_{F_2}\eta_{F_1}^{-1}(\Gamma_C F_1) = (\Gamma_C F_2)\sigma_{F_2}\sigma_{F_1}^{-1} \\
\implies &\alpha_\eta(\Gamma_C F_1) = (\Gamma_C F_2)\alpha_\sigma.
\end{aligned}$$

Thus, $\Gamma_C : \sigma_C \to \eta_C$ is a natural transformation of $\mathscr{A}$–balanced functors, and we have concluded the proof that $I_{\mathscr{C}}$ is well-defined.

$I_{\mathscr{C}}$ is surjective

Let $F : \mathscr{M} \times \mathscr{N} \to \mathscr{C}$ be an $\mathscr{A}$–balanced functor with balancing α i.e. F is an object of

$\mathbf{Fun}_{\mathscr{A}-\mathrm{bal}}(\mathscr{M},\mathscr{N};\mathscr{C})$. Define

$$\sigma = \begin{pmatrix} \sigma_A : A \to \mathscr{C} \text{ is } F \\ \sigma_B : B \to \mathscr{C} \text{ is } FF_1 \\ \sigma_C : C \to \mathscr{C} \text{ is } FF_1G_2 \\ \sigma_{F_1} : FF_1 \to FF_2 \text{ is the identity} \\ \sigma_{F_2} : FF_1 \to FF_2 \text{ is } \alpha \\ \sigma_{G_1} : FF_1G_2 \to FF_1G_1 \text{ is } F\kappa_2^{-1} \\ \sigma_{G_2} : FF_1G_2 \to FF_1G_2 \text{ is the identity} \\ \sigma_{G_3} : FF_1G_2 \to FF_1G_3 \text{ is } (\alpha^{-1}G_3)(F\kappa_3^{-1})(\alpha G_2) \end{pmatrix} \tag{4.1}$$

If σ is a well-defined element of $\mathbf{Diag}_{\mathscr{D}}(\mathbf{Cat}_k)(X, \mathrm{Tr}_q(\mathscr{C}))$ then $I_{\mathscr{C}}(\sigma) = F$, so it remains to show that $\kappa_1 = \Delta_{13}\Delta_{21}^{-1}$, $\sigma_C\kappa_2 = \Delta_{12}\Delta_{11}^{-1}$ and $\sigma_C\kappa_3 = \Delta_{22}\Delta_{23}^{-1}$ where $\Delta_{ij} := (\sigma_{F_i}G_j)\sigma_{G_j}$:

$$\begin{aligned} \Delta_{13}\Delta_{21}^{-1} &= (\sigma_{F_1}G_2)\sigma_{G_2}\sigma_{G_1}^{-1}(\sigma_{F_1}^{-1}G_1) \text{ by definition of } \Delta_{ij} \\ &= F\kappa_2 \text{ by definition of } \sigma_{F_i}, \sigma_{G_j}. \\ \Delta_{22}\Delta_{23}^{-1} &= (\sigma_{F_2}G_2)\sigma_{G_2}\sigma_{G_3}^{-1}(\sigma_{F_2}^{-1}G_3) \text{ by definition of } \Delta_{ij} \\ &= (\alpha G_2)(\alpha^{-1}G_2)(F\kappa_3)(\alpha G_3)(\alpha^{-1}G_3) \text{ by definition of } \sigma_{F_i}, \sigma_{G_j}. \\ &= F\kappa_3. \\ \Delta_{13}\Delta_{21}^{-1} &= (\sigma_{F_1}G_3)\sigma_{G_3}\sigma_{G_1}^{-1}(\sigma_{F_2}^{-1}G_1) \text{ by definition of } \Delta_{ij} \\ &= (\alpha^{-1}G_3)(F\kappa_3^{-1})(\alpha G_2)(F\kappa_2)(\alpha^{-1}G_1) \text{ by definition of } \sigma_{F_i}, \sigma_{G_j} \\ &= F\kappa_1 \text{ i.e. identity, by pentagon of } \alpha. \end{aligned}$$

$I_{\mathscr{C}}$ **is full and faithful.**

Suppose $I_{\mathscr{C}}(\Gamma) = I_{\mathscr{C}}(\Xi)$. By definition of $I_{\mathscr{C}}$ this is $\Gamma = \Xi$; hence, $I_{\mathscr{C}}$ is faithful.

Let $\xi : F \Rightarrow G$ be a $\mathscr{A}$–balanced natural transformation between the $\mathscr{A}$–balanced functors $F, G : \mathscr{M} \times \mathscr{N} \to \mathscr{A}$, i.e. ξ is a morphism of $\mathbf{Fun}_{\mathscr{A}-\mathrm{bal}}(\mathscr{M},\mathscr{N};\mathscr{C})$. We have already shown $I_{\mathscr{C}}$ to be surjective, so we have σ and η such that $I_{\mathscr{C}}(\sigma) = F$ and $I_{\mathscr{C}}(\eta) = G$ where σ is defined in 4.1 and η is defined analogously. In order to show that $I_{\mathscr{C}}$ is full we must find a morphism $\Gamma : \sigma \to \mu$ in $\mathbf{Diag}_{\mathscr{D}}(\mathbf{Cat}_k)(X, \mathrm{Tr}_q(\mathscr{C}))$ such that $I_{\mathscr{C}}(\Gamma) = \xi$.

Define

$$\Gamma = \left(\begin{tikzcd} & \mathscr{M} \times \mathscr{N} \arrow[dl, bend right, "\sigma_C"'] \arrow[dr, bend left, "\eta_C"] \arrow[d, phantom, "\xRightarrow{\Gamma_C := \xi}"] & \\ & \mathscr{C} & \end{tikzcd} \right).$$

As $I_{\mathscr{C}}(\Gamma) = \xi$, it remains to show Γ is a well-defined morphism in $\mathbf{Diag}_{\mathscr{D}}(\mathbf{Cat}_k)(X, \mathrm{Tr}_q(\mathscr{C}))$ i.e. that

1. $\eta_{F_1}^{-1}(\Gamma_C F_1)\sigma_{F_1} = \eta_{F_2}^{-1}(\Gamma_C F_2)\sigma_{F_2}$ and

2. $(\Delta_{ij}^{\eta})^{-1}(\Gamma_C F_i G_j)\Delta_{ij}^{\sigma} = (\Delta_{kl}^{\eta})^{-1}(\Gamma_C F_k G_l)\Delta_{kl}^{\sigma}$ for all $i, k = 1, 2; j, l = 1, 2, 3$.

1. As ξ is an $\mathscr{A}$–balanced natural transformation

$$(\eta_{F_2}\eta_{F_1}^{-1})_{m,a,n}\xi_{(m\lhd a,n)} = \xi_{(m,a\rhd n)}(\sigma_{F_2}\sigma_{F_1}^{-1})_{m,a,n}$$

where $\sigma_{F_2}\sigma_{F_1}^{-1}$ is the balancing of $I_{\mathscr{C}}(\sigma)$ and $\eta_{F_2}\eta_{F_1}^{-1}$ is the balancing of $I_{\mathscr{C}}(\eta)$

$$\implies (\eta_{F_1}^{-1})_{m,a,n}\xi_{(m\lhd a,n)}(\sigma_{F_1})_{m,a,n} = (\eta_{F_2}^{-1})_{m,a,n}\xi_{(m,a\rhd n)}(\sigma_{F_2})_{m,a,n}$$

$$\Longrightarrow \eta_{F_1}^{-1}(\Gamma_C F_1)\sigma_{F_1}(m,a,n) = \eta_{F_2}^{-1}(\Gamma_C F_2)\sigma_{F_2}(m,a,n) \quad \forall (m,a,n) \in \mathscr{M}\times\mathscr{A}\times\mathscr{N}$$
$$\Longrightarrow \eta_{F_1}^{-1}(\Gamma_C F_1)\sigma_{F_1} = \eta_{F_2}^{-1}(\Gamma_C F_2)\sigma_{F_2}$$

2. Denote $\mathrm{Eq}(i,j) := (\Delta^{\eta}_{ij})^{-1}(\Gamma_C F_i G_j)\Delta^{\sigma}_{ij}$. By definition of $\Delta_{i,j}$, σ and η we have that

$$\begin{aligned}
\Delta^{\sigma}_{1j} &= (\sigma_{F_1}G_j)\sigma_{G_j} = \sigma_{G_j}\\
\Delta^{\eta}_{1j} &= (\eta_{F_i}G_j)\eta_{G_j} = \eta_{G_j}\\
\Delta^{\sigma}_{2j} &= (\sigma_{F_1}G_j)\sigma_{G_j} = (\alpha_\sigma G_j)\sigma_{G_j}\\
\Delta^{\eta}_{2j} &= (\eta_{F_1}G_j)\eta_{G_j} = (\alpha_\eta G_j)\eta_{G_j}
\end{aligned}$$

So

$$\begin{aligned}
\mathrm{Eq}(1,j) &= \eta_{G_j}^{-1}(\xi F_1 G_j)\sigma_{G_j} \text{ and}\\
\mathrm{Eq}(2,j) &= \eta_{G_j}^{-1}(\alpha_\eta^{-1}G_j)(\xi F_2 G_j)(\alpha_\sigma G_j)\sigma_{G_j}\\
&= \eta_{G_j}^{-1}\left(\left(\alpha_\eta^{-1}(\xi F_2)\alpha_\sigma\right)G_j\right)\sigma_{G_j}\\
&= \eta_{G_j}^{-1}\left(\xi F_1 G_j\right)\sigma_{G_j} \text{ as } \xi \text{ is } \mathscr{A}\text{–balanced}\\
&= \mathrm{Eq}(1,j).
\end{aligned}$$

This means it remains to show $\mathrm{Eq}(1,1) = \mathrm{Eq}(1,2) = \mathrm{Eq}(1,3)$:

$$\begin{aligned}
\mathrm{Eq}(1,2) &= \eta_{G_2}^{-1}(\xi F_1 G_2)\\
&= (\xi F_1 G_2)\\
\mathrm{Eq}(1,1) &= \eta_{G_1}^{-1}(\xi F_1 G_1)\\
&= (\eta_C \kappa_2)(\xi F_1 G_1)(\sigma_C \kappa_2^{-1})
\end{aligned}$$

$$= \begin{tikzcd}
\mathscr{M}\times\mathscr{A}\times\mathscr{A}\times\mathscr{N} \arrow[r, bend left=70, "F_1G_2"{name=U}] \arrow[r, "F_1G_1"{name=M, description}] \arrow[r, bend right=70, "F_1G_2"'{name=D}] & \mathscr{M}\times\mathscr{N} \arrow[r, bend left=50, "\sigma_C"{name=S}] \arrow[r, bend right=50, "\eta_C"'{name=E}] & \mathscr{C}
\arrow[Rightarrow, from=U, to=M, "\kappa_2^{-1}"]
\arrow[Rightarrow, from=M, to=D, "\kappa_2"]
\arrow[Rightarrow, from=S, to=E, "\xi"]
\end{tikzcd}$$

$$= \begin{tikzcd}
\mathscr{M}\times\mathscr{A}\times\mathscr{A}\times\mathscr{N} \arrow[r, "F_1G_2"] & \mathscr{M}\times\mathscr{N} \arrow[r, bend left=50, "\sigma_C"{name=S}] \arrow[r, bend right=50, "\eta_C"'{name=E}] & \mathscr{C}
\arrow[Rightarrow, from=S, to=E, "\xi"]
\end{tikzcd}$$

$$\begin{aligned}
&= (\xi F_1 G_2)\\
&= \mathrm{Eq}(1,2)\\
\mathrm{Eq}(1,3) &= (\alpha_\eta^{-1}G_2)(\eta_C\kappa_3)(\alpha_\eta G_3)(\xi F_1 G_3)(\alpha_\sigma^{-1}G_3)(\sigma_C\kappa_3^{-1})(\alpha_\sigma G_2)\\
&= (\alpha_\eta^{-1}G_2)(\eta_C\kappa_3)\left(\left(\alpha_\eta(\xi F_1)(\alpha_\sigma^{-1}\right)G_3\right)(\sigma_C\kappa_3^{-1})(\alpha_\sigma G_2)\\
&= (\alpha_\eta^{-1}G_2)(\eta_C\kappa_3)(\xi F_2 G_3)(\sigma_C\kappa_3^{-1})(\alpha_\sigma G_2) \text{ as } \xi \text{ is } \mathscr{A}\text{–balanced}\\
&= (\alpha_\eta^{-1}G_2)(\xi F_2 G_2)(\alpha_\sigma G_2) \text{ by } (\dagger)\\
&= \xi F_1 G_2 \text{ as } \xi \text{ is } \mathscr{A}\text{–balanced}
\end{aligned}$$

$$= \mathrm{Eq}(1,2)$$

where (†):

$$(\eta_C\kappa_3)(\xi F_2G_3)(\sigma_C\kappa_3^{-1}) = \begin{tikzcd} \mathscr{M}\times\mathscr{A}\times\mathscr{A}\times\mathscr{N} \arrow[r, bend left=60, "F_2G_2"{name=U}] \arrow[r, "F_2G_3"{name=M, description}] \arrow[r, bend right=60, "F_2G_2"'{name=D}] \arrow[Rightarrow, from=U, to=M, "\kappa_3^{-1}"] \arrow[Rightarrow, from=M, to=D, "\kappa_3"] & \mathscr{M}\times\mathscr{N} \arrow[r, bend left, "\sigma_C"{name=S}] \arrow[r, bend right, "\eta_C"'{name=E}] \arrow[Rightarrow, from=S, to=E, "\xi"] & \mathscr{C} \end{tikzcd}$$

$$= \begin{tikzcd} \mathscr{M}\times\mathscr{A}\times\mathscr{A}\times\mathscr{N} \arrow[r, "F_2G_2"] & \mathscr{M}\times\mathscr{N} \arrow[r, bend left, "\sigma_C"{name=S}] \arrow[r, bend right, "\eta_C"'{name=E}] \arrow[Rightarrow, from=S, to=E, "\xi"] & \mathscr{C} \end{tikzcd}$$

$$= \xi F_2G_2$$

□

4.2 Skein Categories

4.2.1 Skein Categories and Coloured Ribbon Graphs of Surfaces

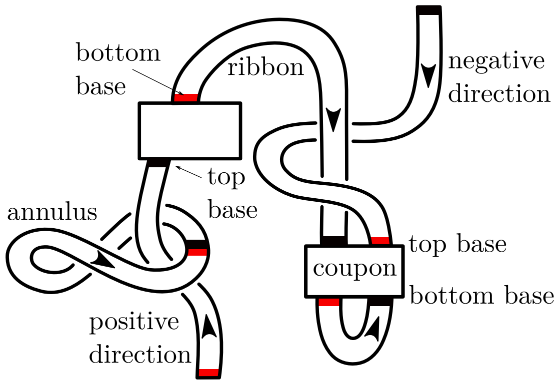
A skein category is a categorical analogue of a skein algebra. The definition we use follows that stated by Johnson-Freyd in [Joh15] which in turn is a modification of a generalisation to a general surfaces of the category $\mathbf{Ribbon}_{\mathscr{V}}$ of coloured ribbon graphs of Reshetikhin and Turaev [RT90, Tur94]. We begin by defining $\mathbf{Ribbon}_{\mathscr{V}}$ for any surface.

Definition 4.2.1. A *ribbon graph* is constructed out of a finite number of ribbons and coupons:

1. A ribbon is a framed strand. The homeomorphic image of $\{0\}$ is called the bottom base of the ribbon and the homeomorphic image of $\{1\}$ is called the top base of the ribbon. Ribbons have two possible directions: up from the bottom base to the top base $(+)$ or down from the top base to the bottom base $(-)$.

2. A *coupon* is homeomorphic to $[0,1]^2$. Its *bases* are the homeomorphic images of $[0,1]\times\{0\}$ and $[0,1]\times\{1\}$: the image of $[0,1]\times\{0\}$ is the bottom base and the image of $[0,1]\times\{1\}$ is the top base.

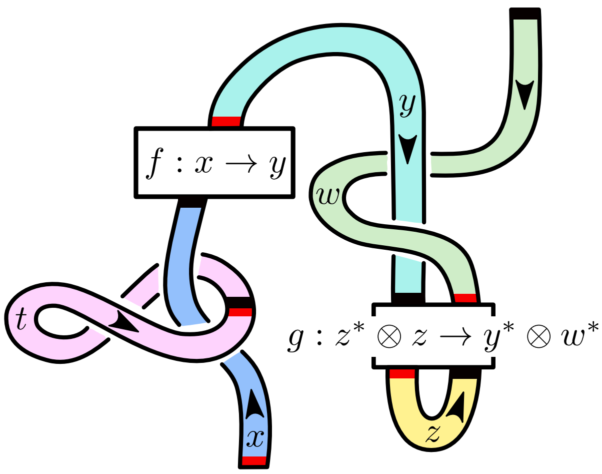
A base of a ribbon may be attached to the base of a coupon, or to the other base of the ribbon to form an annulus, otherwise ribbons and coupons are disjoint.



Definition 4.2.2. Fix a strict monoidal category $\mathscr{V}$. A ribbon graph is coloured by $\mathscr{V}$ as follows:

1. Each ribbon is coloured with a object of $\mathscr{V}$.

2. For a coupon, let $V_1, \ldots, V_n$ and $\epsilon_1, \ldots, \epsilon_n$ be the colours and directions of the strands attached to the bottom base the coupon and let $W_1, \ldots, W_m$ and $\eta_1, \ldots, \eta_m$ be the colours and directions of the strands attached to the top base the coupon—the order in which the bands are attached to base $[0,1]$ gives the ordering. The coupon is coloured by a morphism $f : V_1^{\epsilon_1} \otimes \cdots \otimes V_n^{\epsilon_n} \to W_1^{\eta_1} \otimes \cdots \otimes W_m^{\eta_m}$ of $\mathscr{V}$ where $X^+ := X$ and $X^- := X^*$ for $X \in \mathscr{V}$.



Definition 4.2.3. A *coloured ribbon diagram of a surface* Σ is an embedding of a coloured ribbon graph into $\Sigma \times [0,1]$ such that unattached bases of ribbons are sent to $\Sigma \times \{0,1\}$ and otherwise the image lies in $\Sigma \times (0,1)$. The coupons must be oriented upwards. We call its intersection with $\Sigma \times \{0\}$ the bottom of the diagram and its intersection with $\Sigma \times \{1\}$ the top of the diagram.

Definition 4.2.4. Two coloured strand diagrams are *isomorphic* if there is a finite sequence of isotopies from one to the other each of which are fixed except in the interior of a 3–ball and preserves the attachments of ribbons and directions.

Definition 4.2.5. Fix a strict ribbon category $\mathscr{V}$ and a surface Σ. The k–linear category of $\mathscr{V}$-coloured strands in Σ is denoted $\mathbf{Ribbon}_{\mathscr{V}}(\Sigma)$.

I. The objects of $\mathbf{Ribbon}_{\mathscr{V}}(\Sigma)$ are finite sets $\left\{x_1^{(V_1,\epsilon_1)},\dots,x_n^{(V_n,\epsilon_n)}\right\}$ of disjoint framed points x_i in Σ coloured by objects $V_i \in \mathscr{V}$ and given a direction $\epsilon \in \{+,-\}$.

II. A morphism $F : \left\{x_1^{(V_1,\epsilon_1)},\dots,x_n^{(V_n,\epsilon_n)}\right\} \to \left\{y_1^{(W_1,\delta_1)},\dots,y_m^{(W_m,\delta_m)}\right\}$ of $\mathbf{Ribbon}_{\mathscr{V}}(\Sigma)$ is a finite linear combination $F = \sum_i \lambda_i F_i$ where $\lambda_i \in k$ and F_i is a $\mathscr{V}$–coloured strand diagram such that bottom of the diagram is $\left\{x_1^{(V_1,\epsilon_1)},\dots,x_n^{(V_n,\epsilon_n)}\right\}$ and the top is $\left\{y_1^{(W_1,\delta_1)},\dots,y_m^{(W_m,\delta_m)}\right\}$†.

III. The identity morphism $\mathrm{Id}_{\left\{x_1^{(V_1,\epsilon_1)},\dots,x_n^{(V_n,\epsilon_n)}\right\}}$ is the ribbon diagram consisting of n ribbons which are fixed in Σ–coordinate and framing (up to isomorphism).

IV. The composition of morphisms $F = \sum_i \lambda_i F_i$ and $G = \sum_j \mu_j G_j$ is $G \circ F = \sum_{i,j} \lambda_i \mu_j G_j \circ F_i$ where $G_j \circ F_i$ is given by stacking coloured strand diagrams then retracting $\Sigma \times [0,2]$ to $\Sigma \times [0,1]$. The strands of F_i attached to the top of its diagram and the strands of G_j attached to the bottom of its diagram are merged.

Remark 4.2.6. Note that an embedding of surfaces $p : \Sigma \to \Pi$ induces a functor $P : \mathbf{Sk}(\Sigma) \to \mathbf{Sk}(\Pi)$ of skein categories which on the object x is defined by $P(x) = p(x)$ and on the morphism $F = \sum_i \lambda_i F_i$ is defined by $P(F_i) = \left(p \times \mathrm{Id}_{[0,1]}\right)(F_i)$.

Remark 4.2.7. When $\Sigma = C \times [0,1]$ for some 1-manifold C the k–category $\mathbf{Ribbon}_{\mathscr{V}}(\Sigma)$ can be equipped with a monoidal structure induced by the embedding

$$I : (C \times [0,1]) \sqcup (C \times [0,1]) \hookrightarrow C \times [0,1]$$

which retracts both copies of $C \times [0,1]$ in the second coordinate and includes them into another copy of $C \times [0,1]$. We shall denote the retractions l and r respectively. The monoidal unit is the set $\{\,\}$.

The monoidal category $\mathbf{Ribbon}_{\mathscr{V}}(C \times [0,1])$ has duals with the dual of an object obtained by flipping directions:

$$\left\{x_1^{(V_1,\epsilon_1)},\dots,x_n^{(V_n,\epsilon_n)}\right\}^* := \left\{x_1^{(V_1,-\epsilon_1)},\dots,x_n^{(V_n,-\epsilon_n)}\right\}.$$

The unit and counit are given by a cap and cup respectively. Equipping $\mathbf{Ribbon}_{\mathscr{V}}(C \times [0,1])$ with a braiding and twist given by crossing ribbons are twisting ribbons (see figures in Section 2.2) makes it into a ribbon category. In particular $\mathbf{Ribbon}_{\mathscr{V}}([0,1]^2)$ is a ribbon category.

Proposition 4.2.8 [Tur94]**.** *Let $\mathscr{V}$ be a strict ribbon category. There is a full surjective ribbon functor*

$$\mathrm{eval} : \mathbf{Ribbon}_{\mathscr{V}}([0,1]^2) \to \mathscr{V}.$$

To define a skein category $\mathbf{Sk}_{\mathscr{V}}(\Sigma)$ of a surface we take the ribbon category of the surface $\mathbf{Ribbon}_{\mathscr{V}}(\Sigma)$ and force it to locally satisfy the relations satisfied in $\mathscr{V}$.

Definition 4.2.9. Let Σ be a surface and $\mathscr{V}$ be a strict ribbon category. The k–linear category $\mathbf{Sk}_{\mathscr{V}}(\Sigma)$ is $\mathbf{Ribbon}_{\mathscr{V}}(\Sigma)$ modulo the following relation on the morphisms of $\mathbf{Ribbon}_{\mathscr{V}}(\Sigma)$. For each orientation preserving embedding

$$E : [0,1]^3 \to \Sigma \times [0,1]$$

†Note that this includes the colouring, framings and directions matching.

we set the morphism $F = \sum_i \lambda_i F_i$ to zero if

1. the only intersection of F_i with the boundary of the cube $E(\partial[0,1]^3)$ are transverse ribbons with the top and bottom edge of the cube;
2. the F_i are equal outside of $E([0,1]^3)$;
3. $\sum_i \lambda_i \operatorname{eval}\left(E^{-1}(F_i \cap E([0,1]^3)\right) = 0$ where eval is the functor from Proposition 4.2.8.

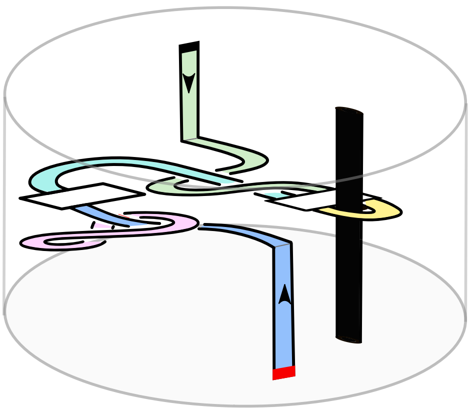
So from Proposition 4.2.8 we conclude

Corollary 4.2.10. *Let $\mathscr{V}$ be a strict ribbon category. Then there is an equivalence of ribbon categories*

$$\mathbf{Sk}_{\mathscr{V}}\left([0,1]^2\right) \simeq \mathscr{V}.$$

Diagrams

The morphisms of a skein category are linear combinations of coloured ribbon diagrams in $\Sigma \times [0,1]$ up to isotopy. A ribbon diagram R can be depicted by diagrams drawn on Σ in a way generalising knot diagrams. Deform the ribbon diagram R so that with the exception of bands attached to end intervals it lies almost parallel and very close to $\Sigma \times \{1/2\}$. The bands attached to end intervals can be deformed so that they do not move in the Σ direction except very close to $\Sigma \times \{1/2\}$. Further deform R so that the coupons of R lie in $\Sigma \times \{1/2\}$, so that no ribbons lie directly above or below (in the t coordinate) coupons, and at most two ribbons lie above or below each other. After having deformed R in this manner we draw the projection of the ribbon diagram onto $\Sigma \times \{1/2\}$ taking account under and over crossings and making start and end intervals as such. The original ribbon diagram R can be recovered up to isotopy from this diagram.



Ribbon tangle diagram in $\Sigma \times [0,1]$

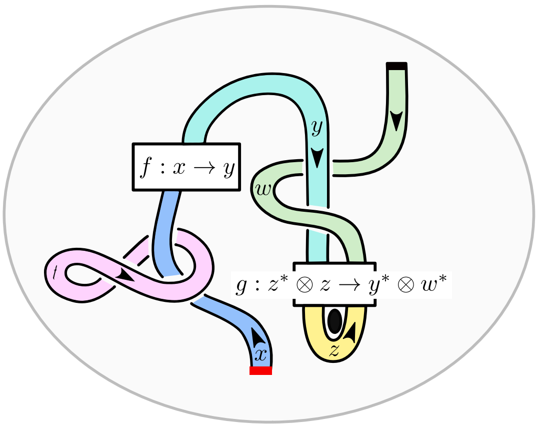


Diagram of the ribbon tangle in Σ

The composition of ribbon diagrams R and S in a skein category with the end points of R equal (in position, framing and colouring) to the start points of S is the ribbon diagram $S \circ R$ formed by placing S above R in $\Sigma \times [0,2]$, gluing the end points of R to the start points of S and deforming $\Sigma \times [0,2]$ to $\Sigma \times [0,1]$. The diagram of $S \circ R$ is given by placing the diagram of S over the diagram of R and removing the start and end points which now join up.

4.2.2 Module Structures and the Relative Tensor Product

We saw in the previous section that $\mathbf{Sk}(C \times [0,1])$ is a monoidal category for any 1-manifold C. Suppose that we have a surface M with boundary ∂M. We shall now show how a suitable embedding of C into ∂M equips $\mathbf{Sk}(M)$ with a $\mathbf{Sk}(C \times [0,1])$–module structure.

Definition 4.2.11. Let C be a 1-manifold and M be a surface with boundary ∂M. A *thickened right embedding* of C into the boundary of M consists of

1. An embedding $\Xi : C \times (-\epsilon, 1] \hookrightarrow M$ such that its restriction to $C \times \{ 1 \}$ gives an embedding $\xi : C \hookrightarrow \partial M$. We denote the restriction $\Phi := \Xi|_{C \times [0,1]}$ and the restriction $\mu := \Xi|_{C \times \{ 0 \}}$.
2. An embedding $E : M \to M$ such that $\mathrm{Im}(E)$ is disjoint from $\mathrm{Im}(\Phi)$.
3. An isotopy $\lambda : M \times [0,1] \to M$ from Id_M to E which is trivial outside of $\mathrm{Im}(\Xi)$.

A *thickened left embedding* is defined similarly except Ξ is an embedding $\Xi : C \times [0, 1+\epsilon) \hookrightarrow M$ such that its restriction to $C \times \{ 0 \}$ gives an embedding $\xi : C \hookrightarrow \partial M$.

Remark 4.2.12. Let $F, G : M \to M$ be two embeddings and let $\sigma : M \times [0,1] \to M$ be an isotopy from F to G. This isotopy traces out for any $m \in \mathbf{Sk}(M)$ a ribbon tangle

$$r_{\sigma,m} : F(m) \to G(m).$$

In particular if (Ξ, E, λ) is a thickened embedding of C into the boundary of M then the isotopy $\lambda : M \times [0,1] \to M$ traces out for any $m \in \mathbf{Sk}(M)$ a ribbon tangle $r_{\lambda,m} : m \to E(m)$. We also have for any $a \in \mathbf{Sk}(C \times [0,1])$ ribbon tangles $r_{l,a} : a \to a * \emptyset$ and $r_{r,a} : a \to \emptyset * a$ where l and r are the retractions used to define the monoidal structure of $\mathbf{Sk}(C \times [0,1]$. Furthermore, for any ribbon tangle $f : m \to m'$ we have that

$$r_{\lambda,m'} \circ f = E(F) \circ r_{\lambda,m}$$

and similarly $r_{l,a}$ and $r_{r,a}$ 'commute' with any ribbon tangle $g : a \to a'$.

Definition 4.2.13. Given a thickened right embedding (Ξ, E, λ) of C into the boundary of M, $\mathbf{Sk}(M)$ is a right $\mathbf{Sk}(C \times [0,1])$–module with action

$$\triangleleft : \mathbf{Sk}(M) \times \mathbf{Sk}(C \times [0,1]) \to \mathbf{Sk}(M)$$

induced from the embedding of surfaces

$$M \sqcup (C \times [0,1]) \to M : M \sqcup A \mapsto E(M) \sqcup \Phi(A).$$

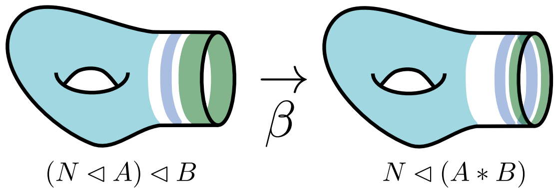
The associator β is defined as

$$\beta_{m,a,b} := r_{\lambda^{-1},(m \triangleleft \emptyset) \triangleleft \emptyset} \sqcup \left(r_{l,\emptyset \triangleleft a} \circ r_{\lambda^{-1},(\emptyset \triangleleft a) \triangleleft a} \right) \sqcup r_{r,(\emptyset \triangleleft \emptyset) \triangleleft b} : (m \triangleleft a) \triangleleft b \to m \triangleleft (a \otimes b)$$

and the unitor η is defined as

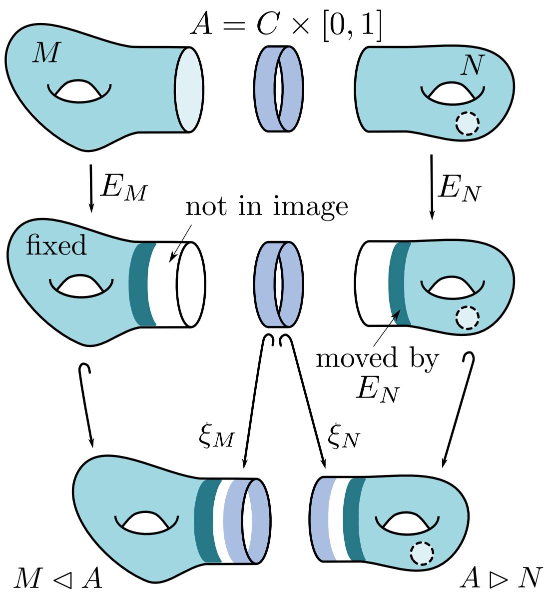
$$\eta_m := r^{-1}_{\lambda,m} : m \triangleleft \emptyset \to m.$$

Analogously, a thickened left embedding (Ξ, E, λ) of C into the boundary of N defines a left $\mathbf{Sk}(C \times [0,1])$–module structure on $\mathbf{Sk}(N)$,



As skein categories are k-linear, we may define the relative tensor product of skein categories to be their relative tensor product as k-linear categories.

Definition 4.2.14. Let C be a 1–manifold with a thickened right embedding (Ξ_M, E_M, λ_M) into the boundary of the surface M and a thickened left embedding (Ξ_N, E_N, λ_N) into the boundary of the surface N. By Definition 4.2.13, $\mathbf{Sk}(M)$ is a right $\mathbf{Sk}(C \times [0,1])$–module and $\mathbf{Sk}(N)$ is a left $\mathbf{Sk}(C \times [0,1])$–module. The *relative tensor product* $\mathbf{Sk}(M) \times_{\mathbf{Sk}(A)} \mathbf{Sk}(M)$ is the relative tensor product as k-linear categories of $\mathbf{Sk}(M)$ and $\mathbf{Sk}(C)$ relative to $\mathbf{Sk}(A)$ (See Definition 4.1.7).



Remark 4.2.15. The simplify notation we shall define $A := C \times [0,1]$.

4.2.3 Excision of Skein Categories

Theorem 4.2.16. *Let C be a 1–manifold with a thickened right embedding (Ξ_M, E_M, λ_M) into the boundary of the surface M and a thickened left embedding (Ξ_N, E_N, λ_N) into the boundary of the surface N. The thickened embeddings define a k–linear functor*

$$F : \mathbf{Sk}(M) \times_{\mathbf{Sk}(A)} \mathbf{Sk}(N) \xrightarrow{\sim} \mathbf{Sk}(M \sqcup_A N)$$

which gives an equivalence of categories, where $\mathbf{Sk}(M) \times_{\mathbf{Sk}(A)} \mathbf{Sk}(N)$ is the relative tensor product category defined in the previous section, and $(M \sqcup_A N)$ is the gluing

$$M \sqcup_A N := M \sqcup N \Big/ \left\{ \xi_N(g,i) \sim \xi_N(g, 1-i) \;\middle|\; g \in \bigsqcup_i \gamma_i, i \in [0,1] \right\}.$$

Before proceeding to the proof of the theorem, we shall define the ribbon tangles $\rho_{m,a,b} \in \mathbf{Sk}(M)$ and $\rho_{a,b,n} \in \mathbf{Sk}(N)$ and prove a couple of identities about them. These will be needed in the proof that F is full and faithful.

Definition 4.2.17. Let $m \in \mathbf{Sk}(M)$ and $a, b \in \mathbf{Sk}(A)$ such that the points in a are disjoint from the points in b. We define the ribbon tangle $\rho_{m,a,b} \in \mathbf{Sk}(M)$ to be

$$\rho_{m,a,b} := r_{\lambda_M, m \lhd a} \sqcup \mathrm{Id}_{\emptyset \lhd b} : m \lhd (a \sqcup b) \to (m \lhd a) \lhd b.$$

Let $n \in \mathbf{Sk}(N)$. We define the ribbon tangle $\rho_{a,b,n} \in \mathbf{Sk}(N)$ to be

$$\rho_{a,b,n} := r_{\lambda_N, b \rhd n} \sqcup \mathrm{Id}_{a \rhd \emptyset} : (a \sqcup b) \rhd n \to a \rhd (b \rhd n)).$$

Lemma 4.2.18. *For any $m \in \mathbf{Sk}(M)$, $n \in \mathbf{Sk}(N)$ and $a, b \in \mathbf{Sk}(A)$ such that the points in a are disjoint from the points in b, we have the identities:*

$$\begin{aligned} \rho_{m,a,b} &= \beta^{-1}_{m,a,b} \circ (\mathrm{Id}_m \lhd (r_{l,a} \sqcup r_{r,b})) \\ \rho_{a,b,n} &= \beta^{-1}_{a,b,n} \circ ((r_{l,a} \sqcup r_{r,b}) \rhd \mathrm{Id}_n) \,. \end{aligned}$$

Proof.

$$\begin{aligned} \beta^{-1}_{m,a,b} \circ (\mathrm{Id}_m \lhd (r_{l,a} \sqcup r_{r,b})) &:= \left(r_{\lambda_M, m \lhd \emptyset} \sqcup \left(r_{\lambda_M, \emptyset \lhd a} \circ r_{l^{-1}, \emptyset \lhd (a * \emptyset)}\right) \sqcup r_{r^{-1}, \emptyset \lhd (\emptyset * b)}\right) \circ (\mathrm{Id}_m \lhd (r_{l,a} \sqcup r_{r,b})) \\ &= r_{\lambda_M, m \lhd \emptyset} \sqcup \left(r_{\lambda_M, \emptyset \lhd a} \circ r_{l^{-1}, \emptyset \lhd (a*\emptyset)} \circ r_{l, \emptyset \lhd a}\right) \sqcup \left(r_{r^{-1}, \emptyset \lhd (\emptyset * b)} \circ r_{r, \emptyset \lhd b}\right) \\ &= r_{\lambda_M, m \lhd a} \sqcup \mathrm{Id}_{\emptyset \lhd b} \\ &= \rho_{a,b,n}. \end{aligned}$$

The other identity is analogous. ☐

Lemma 4.2.19. *For any $m \in \mathbf{Sk}(M)$, $n \in \mathbf{Sk}(N)$ and $a, b \in \mathbf{Sk}(A)$ such that the points in a are disjoint from the points in b, the following diagram commutes.*

$$\begin{tikzcd} & (m \lhd a, b \rhd n) \arrow[dr, "\iota_{m,a,b \rhd n}"] & \\ ((m \lhd a) \lhd b, n) \arrow[ur, "\iota_{m \lhd a, b, n}"] & & (m, a \rhd (b \rhd n)) \\ (m \lhd (a \sqcup b), n) \arrow[u, "(\rho_{m,a,b}, \mathrm{Id}_n)"] \arrow[rr, "\iota_{m, a \sqcup b, n}"] & & (m, (a \sqcup b) \rhd n) \arrow[u, "(\mathrm{Id}_m, \rho_{a,b,n})"'] \end{tikzcd}$$

We shall refer to this diagram as the pentagon.

Proof.

$$\begin{tikzcd}
 & (m \lhd a, b \rhd n) \arrow[dr, "\iota_{m,a,b \rhd n}"] & \\
((m \lhd a) \lhd b, n) \arrow[ur, "\iota_{m \lhd a,b,n}"] & \text{pentagon of } \beta & (m, a \rhd (b \rhd n)) \\
(m \lhd (a * b), n) \arrow[u, "(\beta^{-1}_{m,a,b}{,}\mathrm{Id}_n)"] \arrow[rr, "\iota_{m,a*b,n}"] & & (m, (a * b) \rhd n) \arrow[u, "(\mathrm{Id}_m{,}\beta^{-1}_{a,b,n})"'] \\
 & \text{naturality of } \iota & \\
(m \lhd (a \sqcup b), n) \arrow[uu, "(\mathrm{Id}_m \lhd (r_{l,a} \sqcup r_{r,b}){,}\mathrm{Id}_n)"] \arrow[rr, "\iota_{m,a \sqcup b,n}"] & & (m, (a \sqcup b) \rhd n) \arrow[uu, "(\mathrm{Id}_m{,}(r_{l,a}{,}r_{r,b}) \rhd \mathrm{Id}_n)"']
\end{tikzcd}$$

$\square$

Lemma 4.2.20. *Let $f : m \to m'$ be a morphism in $\mathbf{Sk}(M)$, $g : a \to a'$ be a morphism in $\mathbf{Sk}(M)$ which is disjoint from Id_b and $h : b \to b'$ be a morphism in $\mathbf{Sk}(M)$ which is disjoint from Id_a. The following diagrams commute:*

$$\begin{tikzcd}
m \lhd (a \sqcup b) \arrow[r, "f \lhd (\mathrm{Id}_a \sqcup h)"] \arrow[d, "\rho_{m,a,b}"] & m' \lhd (a \sqcup b') \arrow[d, "\rho_{m',a,b'}"] \\
(m \lhd a) \lhd b \arrow[r, "(f \lhd \mathrm{Id}_a) \lhd h"] & (m' \lhd a) \lhd b'
\end{tikzcd}$$

$$\begin{tikzcd}
m \lhd (a \sqcup b) \arrow[r, "f \lhd (g \sqcup \mathrm{Id}_b)"] \arrow[d, "\rho_{m,a,b}"] & m' \lhd (a' \sqcup b) \arrow[d, "\rho_{m',a,b'}"] \\
(m \lhd a) \lhd b \arrow[r, "(f \lhd g) \lhd \mathrm{Id}_b"] & (m' \lhd a') \lhd b
\end{tikzcd}$$

We have a similar result for the ρ in $\mathbf{Sk}(N)$. We shall refer to this as the naturality of ρ.

Proof. This follows from the similar naturality of r_{λ_M} and r_{λ_N}. $\square$

We now proceed to the proof of excision.

Proof of Theorem 4.2.16. We shall first define

$$F : \mathbf{Sk}(M) \times_{\mathbf{Sk}(A)} \mathbf{Sk}(N) \to \mathbf{Sk}(M \sqcup_A N)$$

and show this definition is well-defined, and then show that F is full, faithful and essentially surjective.

Definition of F

We began by defining F:

Objects: Let (m, n) be an object of $\mathbf{Sk}(M) \times_{\mathbf{Sk}(A)} \mathbf{Sk}(N)$, so m is a finite set of disjoint framed directed coloured points in M and n is a finite set of disjoint framed directed coloured points in N. We define

$$F(m, n) := E_M(m) \sqcup E_N(n)$$

which is a finite set of disjoint framed directed coloured points in $M \sqcup_A N$, and thus is a object of $\mathbf{Sk}(M \sqcup_A N)$.

Morphisms: By the definition of the relative tensor product (Definition 4.1.7), the morphisms of $\mathbf{Sk}(M) \times_{\mathbf{Sk}(A)} \mathbf{Sk}(N)$ are generated by the morphisms

1. $(f, g) : (m, n) \to (m', n')$, where $f \in \mathrm{Hom}_{\mathbf{Sk}(M)}(m, m')$ and $g \in \mathrm{Hom}_{\mathbf{Sk}(N)}(n, n')$,
2. $\iota_{m,a,n} : (m \triangleleft a, n) \to (m, a \triangleright n)$ for $(m, a, n) \in \mathbf{Sk}(M) \times \mathbf{Sk}(A) \times \mathbf{Sk}(N)$, and
3. $\iota^{-1}_{m,a,n} : (m, a \triangleright n) \to (m \triangleleft a, n)$ for $(m, a, n) \in \mathbf{Sk}(M) \times \mathbf{Sk}(A) \times \mathbf{Sk}(N)$,

so to define F it suffices to define F for these morphisms:

1. $F(f, g) := E_M(f) \sqcup E_N(g) \in \mathrm{Hom}_{\mathbf{Sk}(M \sqcup_A N)}(E_M(m) \sqcup E_N(n), E_M(m') \sqcup E_N(n'))$ where E is the functor of categories induced by the embedding E.
2. $F(\iota_{m,a,n}) := r^{-1}_{\lambda_M, E^2_M(m)} \sqcup \left(r_{\lambda_N,a} \circ r^{-1}_{\lambda_M, E_M(a)}\right) \sqcup r_{\lambda_N, E(n)} \in \mathrm{Hom}_{\mathbf{Sk}(M \sqcup_A N)}(E^2_M(m) \sqcup E_M(a) \sqcup E_N(n), E_M(m) \sqcup E_N(a) \sqcup E^2_N(n))$
3. $F(\iota^{-1}_{m,a,n}) := r_{\lambda_M, E_M(m)} \sqcup \left(r_{\lambda_M,a} \circ r^{-1}_{\lambda_N, E_N(a)}\right) \sqcup r^{-1}_{\lambda_N, E_N(n)} \in \mathrm{Hom}_{\mathbf{Sk}(M \sqcup_A N)}(E_M(m) \sqcup E_N(a) \sqcup E^2_N(n), E^2_M(m) \sqcup E_M(a) \sqcup E_N(n))$

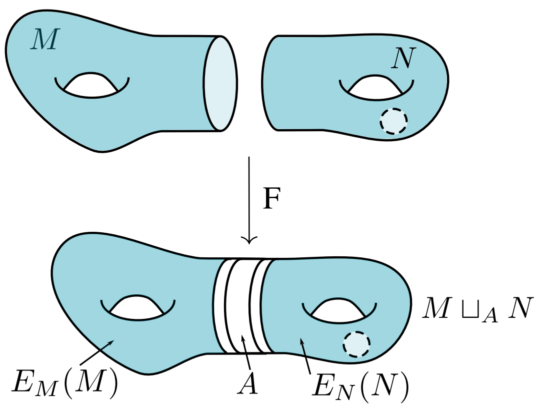


Figure 4.1: This embedding of surfaces induces a functor $\mathbf{Sk}(M) \times \mathbf{Sk}(N) \to \mathbf{Sk}(M \sqcup_A N)$ of their skein categories. The functor F on $P(\mathbf{Sk}(M) \times \mathbf{Sk}(N))$ is given by this functor: that is on objects and on morphisms of the form (f, g).

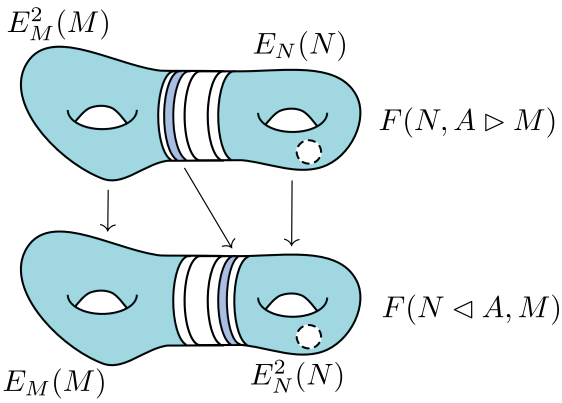


Figure 4.2: The functor F on the natural isomorphism ι gives a ribbon which has stands which cross the middle section from $F(\emptyset \triangleleft a, \emptyset)$ to $F(\emptyset, a \triangleright \emptyset)$ (coloured red). Elsewhere applying $F(\iota_{m,a,n})$ only moves points a little.

In order to show that F is well-defined we must show $F(\textit{morphism})$ still satisfies the relations in Definition 4.1.7. This is a sequence of straight forward calculations:

Linearity Follows automatically as we have defined F to be k–linear.

Functionality Follows from the functionality of the functors E_M and E_N:

$$\begin{aligned}F((f',g')\circ(f,g)) &= E_M(f'\circ f)\sqcup E_N(g'\circ g)\\ &= (E_M(f')\circ E_M(f))\sqcup(E_N(g')\circ E_N(g))\\ &= F((f'\circ f,g'\circ g))\\ F(\mathrm{Id}_{m,n}) &= (E_M(\mathrm{Id}_m),E_N(\mathrm{Id}_n)) = (\mathrm{Id}_{E_M(m)},\mathrm{Id}_{E_N(n)}) = \mathrm{Id}_{F(m,n)}\end{aligned}$$

Isomorphism Follows directly from the definitions:

$$\begin{aligned}F(\iota_{m,a,n})\circ F(\iota^{-1}_{m,a,n}) &:= \Big(r^{-1}_{\lambda_M,E^2_M(m)}\sqcup\Big(r_{\lambda_N,a}\circ r^{-1}_{\lambda_M,E_M(a)}\Big)\sqcup r_{\lambda_N,E(n)}\Big)\\ &\quad\circ\Big(r_{\lambda_M,E_M(m)}\sqcup\Big(r_{\lambda_M,a}\circ r^{-1}_{\lambda_N,E_N(a)}\Big)\sqcup r^{-1}_{\lambda_N,E^2_N(n)}\Big)\\ &= \mathrm{Id}_{E_M(m)}\sqcup\mathrm{Id}_{E_N(a)}\sqcup\mathrm{Id}_{E^2_N(n)}\\ &= \mathrm{Id}_{E(m)\sqcup E(a\rhd n)}\end{aligned}$$

and similarly for $F(\iota^{-1}_{m,a,n})\circ F(\iota_{m,a,n})$.

Naturality This follows from Remark 4.2.12:

$$\begin{aligned}F(\iota_{m',a',n'})\circ F(f\lhd g,h) &= \Big(r^{-1}_{\lambda_M,E^2_M(m')}\sqcup\Big(r_{\lambda_N,a'}\circ r^{-1}_{\lambda_M,E_M(a')}\Big)\sqcup r_{\lambda_N,E_N(n')}\Big)\circ\big(E^2_M(f)\sqcup E_M(g)\sqcup E_N(h)\big)\\ &= \Big(r^{-1}_{\lambda_M,E^2_M(m')}\circ E^2_M(f)\Big)\sqcup\Big(r_{\lambda_N,a'}\circ r^{-1}_{\lambda_M,E_M(a')}\circ E_M(g)\Big)\sqcup\big(r_{\lambda_N,E(n')}\circ E(h)\big)\\ &= \Big(E_M(f)\circ r^{-1}_{\lambda_M,E^2_M(m)}\Big)\sqcup\Big(E_N(g)\circ r_{\lambda_N,a}\circ r^{-1}_{\lambda_M,E_M(a)}\Big)\sqcup\big(E^2(h)\circ r_{\lambda_N,E(n)}\big)\\ &= F(f,g\rhd h)\circ F(\iota_{m,a,n})\end{aligned}$$

Triangle Follows from the definitions:

$$\begin{aligned}F(\mathrm{Id}_m,\eta_n)\circ F(\iota_{m,\emptyset,n}) &= \Big(\mathrm{Id}_{E(m)}\sqcup r^{-1}_{\lambda_N,E_N(n)}\Big)\circ\Big(r^{-1}_{\lambda_M,E^2_M(m)}\sqcup r_{\lambda_N,E_N(n)}\Big)\\ &= r^{-1}_{\lambda_M,E^2(m)}\sqcup\mathrm{Id}_{E_N(n)}\\ &= F(\theta_{m\lhd\emptyset},\mathrm{Id}_n)\end{aligned}$$

Pentagon As

$$\begin{aligned}\beta_{a,b,n} &= r_{\lambda_N^{-1},\emptyset\rhd(\emptyset\rhd n)}\sqcup\Big(r_{r,b\rhd\emptyset}\circ r_{\lambda_N^{-1},\emptyset\rhd(b\rhd\emptyset)}\Big)\sqcup r_{l,a\rhd\emptyset}\\ \beta^{-1}_{m,a,b} &= r_{\lambda_M,m\lhd\emptyset}\sqcup\big(r_{\lambda_M,\emptyset\lhd a}\circ r_{l^{-1},\emptyset\lhd(a*\emptyset)}\big)\sqcup r_{r^{-1},\emptyset\lhd(\emptyset*b)}\end{aligned}$$

we have that

$$\begin{aligned}F((\beta^{-1}_{m,a,b},\mathrm{Id}_n) &= r_{\lambda_M,E^2_M(m)}\sqcup\big(r_{\lambda_M,E_M(a)}\circ r_{l^{-1},E_M(a*\emptyset)}\big)\sqcup r_{r^{-1},E_M(\emptyset*b)}\sqcup\mathrm{Id}_{E_N(n)}\\ F(\mathrm{Id}_m,\beta_{a,b,n}) &= \mathrm{Id}_{E_M(m)}\sqcup r_{\lambda_N^{-1},E^3_N(n)}\sqcup\Big(r_{r,E_N(b)}\circ r_{\lambda_N^{-1},E^2_N(b)}\Big)\sqcup r_{l,E_N(a)}\end{aligned}$$

So,

$$F(\mathrm{Id}_m,\beta_{a,b,n})\circ F(\iota_{m,a,b\rhd n})\circ F(\iota_{m\lhd a,b,n})\circ F(\beta^{-1}_{m,a,b},\mathrm{Id}_n)$$

$$
\begin{aligned}
&= \Big(\mathrm{Id}_{E_M(m)} \sqcup r_{\lambda_N^{-1},E_N^3(n)} \sqcup \Big(r_{r,E_N(b)} \circ r_{\lambda_N^{-1},E_N^2(b)}\Big) \sqcup r_{l,E_N(a)}\Big)\\
&\quad \circ \Big(r_{\lambda_M^{-1},E_M^2(m)} \sqcup \Big(r_{\lambda_N,a} \circ r_{\lambda_M^{-1},E_M(a)}\Big) \sqcup r_{\lambda_N,E(b\triangleright n)}\Big)\\
&\quad \circ \Big(r_{\lambda_M^{-1},E_M^2(m\triangleleft a)} \sqcup \Big(r_{\lambda_N,b} \circ r_{\lambda_M^{-1},E_M(b)}\Big) \sqcup r_{\lambda_N,E(n)}\Big)\\
&\quad \circ \Big(r_{\lambda_M,E_M^2(m)} \sqcup \big(r_{\lambda_M,E_M(a)} \circ r_{l^{-1},E_M(a*\emptyset)}\big) \sqcup r_{r^{-1},E_M(\emptyset*b)} \sqcup \mathrm{Id}_{E_N(n)}\Big)\\
&= \Big(\mathrm{Id}_{E_M(m)} \circ r_{\lambda_M^{-1},E_M^2(m)} \circ r_{\lambda_M^{-1},E_M^3(m)} \circ r_{\lambda_M,E_M^2(m)}\Big)\\
&\quad \sqcup \Big(r_{l,E_N(a)} \circ r_{\lambda_N,a} \circ r_{\lambda_M^{-1},E_M(a)} \circ r_{\lambda_M^{-1},E_M^2(a)} \circ r_{\lambda_M,E_M(a)} \circ r_{l^{-1},E_M(a*\emptyset)}\Big)\\
&\quad \sqcup \Big(r_{r,E_N(b)} \circ r_{\lambda_N^{-1},E_N^2(b)} \circ r_{\lambda_N,E(b)} \circ r_{\lambda_N,b} \circ r_{\lambda_M^{-1},E_M(b)} \circ r_{r^{-1},E_M(\emptyset*b)}\Big)\\
&\quad \sqcup \Big(r_{\lambda_N^{-1},E_N^3(n)} \circ r_{\lambda_N,E^2(n)} \circ r_{\lambda_N,E(n)} \circ \mathrm{Id}_{E_N(n)}\Big)\\
&= \Big(r_{\lambda_M^{-1},E_M^2(m)}\Big) \sqcup \Big(r_{l,E_N(a)} \circ r_{\lambda_N,a} \circ r_{\lambda_M^{-1},E_M(a)} \circ r_{l^{-1},E_M(a*\emptyset)}\Big)\\
&\quad \sqcup \Big(r_{r,E_N(b)} \circ r_{\lambda_N,b} \circ r_{\lambda_M^{-1},E_M(b)} \circ r_{r^{-1},E_M(\emptyset*b)}\Big) \sqcup \big(r_{\lambda_N,E(n)}\big)\\
&= \Big(r_{\lambda_M^{-1},E_M^2(m)}\Big) \sqcup \Big(r_{\lambda_N,a*\emptyset} \circ r_{\lambda_M^{-1},E_M(a*\emptyset)}\Big) \sqcup \Big(r_{\lambda_N,\emptyset*b} \circ r_{\lambda_M^{-1},E_M(\emptyset*b)}\Big) \sqcup \big(r_{\lambda_N,E(n)}\big)\\
&= F(\iota_{m,a*b,n})
\end{aligned}
$$

Remark 4.2.21. These identities have straightforward interpretations topologically, for example the pentagon identity holds as one can straighten strands.

F is essentially surjective

Any point in $E_M(M) \sqcup E_N(N) \subset M \sqcup_A N$ is in the image of F. If point $x^{(V,\epsilon)}$ is not in this region then there is a ribbon which translates $x^{(V,\epsilon)}$ across the middle region to a point $\tilde{x}^{(V,\epsilon)}$ which is in this region. Hence, every point in $M \sqcup_A N$ is isomorphic to an point in the image of F, and F is essentially surjective.

F is full

Let $(m_1, n_1), (m_2, n_2)$ be any objects in $\mathbf{Sk}(M) \times_{\mathbf{Sk}(A)} \mathbf{Sk}(N)$ and let

$$[\overline{u}] \in \mathrm{Hom}_{\mathbf{Sk}(M\sqcup_A N)}\left(F(m_1, n_1), F(m_2, n_2)\right),$$

so $[\overline{u}]$ is the equivalence class of a ribbon diagram

$$\overline{u} : E_M(m_1) \sqcup E_N(n_1) \to E_M(m_2) \sqcup E_N(n_2).$$

In order to show F is full, we must show there is a morphism

$$w \in \mathrm{Hom}_{\mathbf{Sk}(M)\times_{\mathbf{Sk}(A)}\mathbf{Sk}(N))}\left((m_1, n_1), (m_2, n_2)\right)$$

such that $F(w) = u$ for some u equivalent to $\overline{u}$.

We shall call $\mathrm{Im}(\Xi_M \cup \Xi_N) \times [0,1]$, the *middle region*. Up to isotopy fixed outside this middle region, we may assume that $\overline{u}$ intersects $\mathrm{Im}(\mu_M) \times [0,1]$ in a finite number of transverse strands. Let $t_i \in [0,1]$ be the levels when u_{t_i} intersects $\mathrm{Im}(\mu_M)$. By an isotopy in the t–coordinate which moves coupons, twists, minima, maxima, and strands not lying in $F(M,N)$[†], we may assume that u_{t_i} consists entirely of framed points in $F(M,N)$. Up to isotopy fixed in the middle region, we can further assume u_{t_i} contains framed points entirely in $F(M \triangleleft A, A \triangleright N)$. This means that $u_{t_i} = \left(m \triangleleft \left(a \sqcup \overline{b}\right), (\overline{c} \sqcup d) \triangleright n\right)$ where only $\overline{b}$ and $\overline{c}$ intersect $\mathrm{Im}(\mu_M) \sqcup \mathrm{Im}(\mu_N)$. We reparametrise further so that for some small $\epsilon_i > 0$, $u_{[t_i-\epsilon_i, t_i+\epsilon]} = \mathrm{Id}_{m \triangleleft a, d \triangleright n} \sqcup v_{[t_i-\epsilon_i,t_i+\epsilon]}$ where $v_{[t-\epsilon,t+\epsilon]} : E_M(b) \sqcup E_N(c) \to E_M(b') \sqcup E_N(c')$: in other words $u_{[t_i-\epsilon_i,t_i+\epsilon]}$ consists of identity strands and a ribbon tangle which straddles the middle region.

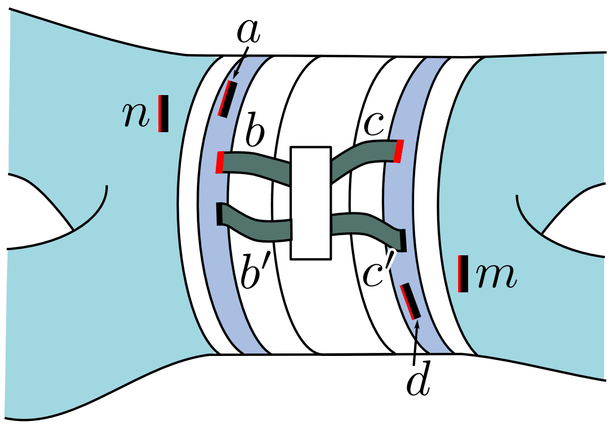


Figure 4.3: An example of $u_{[t_i-\epsilon_i,t_i+\epsilon]}$. In general $a, b, b', \dots$ are not single framed points, but finite sets of framed points, and the coupon depicted could be any ribbon diagram in this square with the same inputs and outputs.

We now have a ribbon diagram u equivalent to $\overline{u}$ with a decomposition

$$u = u_{[1,t_N+\epsilon_N]} \circ u_{[t_N-\epsilon_N,t_N+\epsilon_N]} \circ u_{t_{N-1}+\epsilon_{N-1},t_N-\epsilon_N} \circ \cdots \circ u_{[t_1-\epsilon_1,t_1+\epsilon_1] \circ u_{[0,t_1-\epsilon_1]}}$$

such that $u_{[t_i-\epsilon_i,t_i+\epsilon_i]} = \mathrm{Id}_{m_i \triangleleft a_i, d_i \triangleright n_i} \sqcup v_{[t_i-\epsilon_i,t_i+\epsilon_i]}$ and the other morphisms in the decomposition lie in $F(M,N) \times [0,1]$. If a morphism lies in $F(M,N) \times [0,1]$ then it is of the form $f \sqcup g$ for $f \in F(M) \times [0,1]$ and $g \in F(N) \times [0,1]$. In which case $F(E_M^{-1}(f), E_N^{-1}(g)) = (f,g)$. So it remains to consider the ribbon tangle

$$u_{[t-\epsilon,t+\epsilon]} = v_{[t-\epsilon,t+\epsilon]} \sqcup \mathrm{Id}_{E_M^2(m) \sqcup E_M(a) \sqcup E_N(d) \sqcup E_N^2(n)} : F(m \triangleleft (a \sqcup b), (c \sqcup d) \triangleright n) \to F(m \triangleleft (a \sqcup b'), (c' \sqcup d) \triangleright n)$$

where $v_{[t-\epsilon,t+\epsilon]} : E_M(b) \sqcup E_N(c) \to E_M(b') \sqcup E_N(c)$. As the middle region is topologically trivial, there exists a ribbon tangle $\overline{v} : b \triangleright c \to b' \triangleright c'$ in $\mathbf{Sk}(M)$ such that

$$v_{[t-\epsilon,t+\epsilon]} = \left(\left(r_{\lambda_M,b'} \circ r^{-1}_{\lambda_N,E_N(b')}\right) \sqcup r_{\lambda_N,E_N(c')}\right) \circ E_N(\overline{v}) \circ \left(\left(r_{\lambda_N,b} \circ r^{-1}_{\lambda_M,E_M(b)}\right) \sqcup r_{\lambda_N^{-1},E_N^2(c)}\right).$$

–

[†]Being able to do this relies on the ribbon diagram u not starting or ending in the middle region.

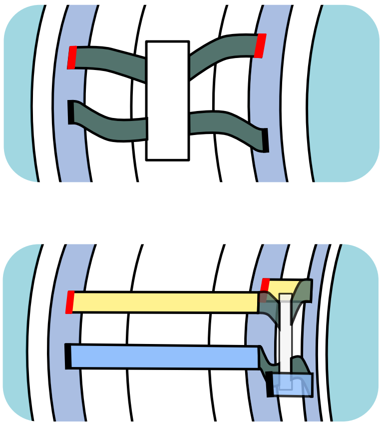


Figure 4.4: The top figure is an example of $v_{[t-\epsilon,t+\epsilon]}$. The bottom figure is isotopic and depicts the decomposition of $v_{[t-\epsilon,t+\epsilon]}$:

1. The yellow ribbons are
$$\left(\left(r_{\lambda_N,b} \circ r^{-1}_{\lambda_M,E_M(b)}\right) \sqcup r_{\lambda_N^{-1},E_N^2(c)}\right);$$
2. The distorted copy of $v_{[t-\epsilon,t+\epsilon]}$ is $\overline{v}$;
3. The blue ribbons are
$$\left(\left(r_{\lambda_M,b'} \circ r^{-1}_{\lambda_N,E_N(b')}\right) \sqcup r_{\lambda_N,E_N(c')}\right).$$

We denote by $w_{[t-\epsilon,t+\epsilon]}$ the following morphism in $\mathbf{Sk}(M) \times_{\mathbf{Sk}(A)} \mathbf{Sk}(N)$:

$$\begin{array}{c}
(m \lhd (a \sqcup b), (c \sqcup d) \rhd n) \\
\Big\downarrow{\scriptstyle (\rho_{m,a,b},\mathrm{Id})} \\
((m \lhd a) \lhd b, (c \sqcup d) \rhd n) \\
\Big\downarrow{\scriptstyle \iota_{m\lhd a,b,(c\sqcup d)\rhd n}} \\
(m \lhd a, b \rhd ((c \sqcup d) \rhd n)) \\
\Big\downarrow{\scriptstyle (\mathrm{Id}_{m\lhd a},\overline{v}\sqcup\mathrm{Id}_{\emptyset\rhd(d\rhd n)})} \\
(m \lhd a, b' \rhd ((c' \sqcup d) \rhd n)) \\
\Big\downarrow{\scriptstyle \iota^{-1}_{m\lhd a,b',(c'\sqcup d)\rhd n}} \\
((m \lhd a) \rhd b', (c' \sqcup d) \rhd n) \\
\Big\downarrow{\scriptstyle (\rho^{-1}_{m,a,b'},\mathrm{Id})} \\
(m \lhd (a \sqcup b'), (c' \sqcup d) \rhd n)
\end{array}$$

We shall sometimes denote $\hat{v} := \overline{v} \sqcup \mathrm{Id}_{\emptyset\rhd(d\rhd n)}$. We claim that $F(w_{[t-\epsilon,t+\epsilon]}) = v_{[t-\epsilon,t+\epsilon]}$. By the functorality of F and the definition of F on the various components, we have that $F(w_{[t-\epsilon,t+\epsilon]})$ is

$$\begin{array}{l}
F(m \lhd (a \sqcup b), (c \sqcup d) \rhd n) \\
\quad\Big\downarrow{\scriptstyle r_{\lambda_M,E_M(m\lhd a)}\sqcup\mathrm{Id}_{E_M(b)} \sqcup \mathrm{Id}_{E_N(c)} \sqcup \mathrm{Id}_{E_N(d\rhd n)}} \\
F((m \lhd a) \lhd b, (c \sqcup d) \rhd n) \\
\quad\Big\downarrow{\scriptstyle r_{\lambda_M^{-1},E_M(m\lhd a)}\sqcup\left(r_{\lambda_N,b}\circ r^{-1}_{\lambda_M,E_M(b)}\right)\sqcup r_{\lambda_N,E_N(c)}\sqcup r_{\lambda_N,E_N(d\rhd n)}} \\
F(m \lhd a, b \rhd ((c \sqcup d) \rhd n)) \\
\quad\Big\downarrow{\scriptstyle \left(\mathrm{Id}_{E_M(m\lhd a)},E_N(\overline{v})\sqcup\mathrm{Id}_{E_N^2(d\rhd n)}\right)} \\
F(m \lhd a, b' \rhd ((c' \sqcup d) \rhd n)) \\
\quad\Big\downarrow{\scriptstyle r_{\lambda_M,E_M(m\lhd a)}\sqcup\left(r_{\lambda_M,b'}\circ r_{\lambda_N^{-1},E_N(b')}\right)\sqcup r^{-1}_{\lambda_N,E_N(c')}\sqcup r^{-1}_{\lambda_N,E_N(d\rhd n)}} \\
F((m \lhd a) \rhd b', (c' \sqcup d) \rhd n) \\
\quad\Big\downarrow{\scriptstyle r_{\lambda_M^{-1},E_M^2(m\lhd a)}\sqcup\mathrm{Id}_{E_M(b)} \sqcup \mathrm{Id}_{E_N(c')} \sqcup \mathrm{Id}_{E_N(d\rhd n)}} \\
F(m \lhd (a \sqcup b'), (c' \sqcup d) \rhd n)
\end{array}$$

So it decomposes into three components:

$$\begin{aligned}
F(w_{[t-\epsilon,t+\epsilon]}) &= \mathrm{Id}_{E_M(m\lhd a)} \sqcup \mathrm{Id}_{E_N(d\rhd n)} \\
&\quad \sqcup \left(\left(r_{\lambda_M,b'} \circ r^{-1}_{\lambda_N,E_N(b')}\right) \sqcup r_{\lambda_N,E_N(c')}\right) \circ E_N(\overline{v}) \circ \left(\left(r_{\lambda_N,b} \circ r^{-1}_{\lambda_M,E_M(b)}\right) \sqcup r_{\lambda_N^{-1},E_N^2(c')}\right) \\
&= \mathrm{Id}_{E_M(m\lhd a)} \sqcup \mathrm{Id}_{E_N(d\rhd n)} \sqcup v \\
&= u_{[t-\epsilon,t+\epsilon]}.
\end{aligned}$$

and we are done.

F is faithful

In the previous section we have shown that for any ribbon tangle u there is a morphism w such that $F(w) = u$. We shall now show that this defines a well defined inverse map of

$$F_{(m_1,n_1),(m_2,n_2)} : \mathrm{Hom}_{\mathbf{Sk}(M)\times_{\mathbf{Sk}(A)}\mathbf{Sk}(N)}\left((m_1,n_1),(m_2,n_2)\right) \to \mathrm{Hom}_{\mathbf{Sk}(M\sqcup_A N)}\left(F(m_1,n_1),F(m_2,n_2)\right).$$

If the map $u \mapsto w$ is well defined it is the inverse of $F_{(m_1,n_1),(m_2,n_2)}$, because $F(f,g) \mapsto (f,g)$ and $F(\iota_{m,a,n}) \mapsto \iota_{m,a,n}$.

Any equivalence of ribbon diagrams in $\mathbf{Sk}(M \sqcup_A N)$ can be decomposed into equivalences which are fixed outside of one open set in the open cover of $(M \sqcup_A N) \times [0,1]$. In particular this means that any isotopy of

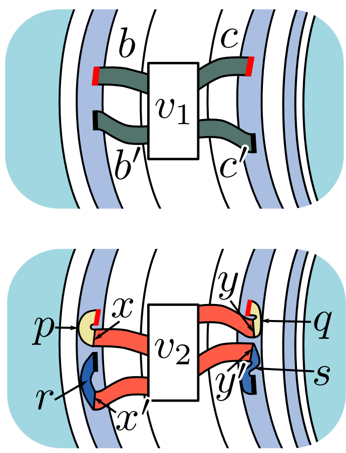
$$u = u_{[1,t_N+\epsilon_N]} \circ u_{[t_N-\epsilon_N,t_N+\epsilon_N]} \circ u_{[t_{N-1}+\epsilon_{N-1},t_N-\epsilon_N]} \circ \cdots \circ u_{[t_1-\epsilon_1,t_1+\epsilon_1]} \circ u_{[0,t_1-\epsilon_1]}$$

consists of the composition of equivalence of the following forms:

1. **Equivalence of a non-crossing morphism.** Let $u_i := u_{[t+\epsilon,t-\epsilon]}$ be a non-crossing ribbon diagram, so $u_i = f \sqcup g$ for $f \in F(M)\times[0,1]$ and $g \in F(N)\times[0,1]$. The equivalences $f \sim f'$ and $g \sim g'$ of the ribbon tangles in $F(M)\times[0,1]$ and $F(N)\times[0,1]$ respectively define an equivalence of u_i to another non-crossing ribbon diagram $u_i' := f' \sqcup g'$.

2. **Equivalence in the middle region**.

   Let $u_i := u_{[t-\epsilon,t+\epsilon]}$ be a crossing ribbon diagram, so $u_i = v_1 \sqcup \mathrm{Id}_{F(m\lhd a,d\rhd n)}$. The equivalence in the middle region $v_1 \sim (r \sqcup s) \circ v_2 \circ (p \sqcup q)$ where $r, p \in F(\emptyset \lhd A)\times[0,1]$ and $s, q \in F(A \rhd \emptyset)\times[0,1]$ depicted in the figure opposite defines an equivalence of u_i to $\left(r \sqcup s \sqcup \mathrm{Id}_{F(m\lhd a,d\rhd n)}\right) \circ \left(v_2 \sqcup \mathrm{Id}_{F(m\lhd a,d\rhd n)}\right) \circ \left(p \sqcup q \sqcup \mathrm{Id}_{F(m\lhd a,d\rhd n)}\right)$.



3. **Commuting with a crossing.** Let $u_i := u_{[s,t-\epsilon]}$[†] be a non-crossing ribbon diagram of the form $u_i = g \sqcup h \sqcup \mathrm{Id}_{F(b,c)}$ where $g \in F(M)\times[0,1]$ and $h \in F(N)\times[0,1]$ and let

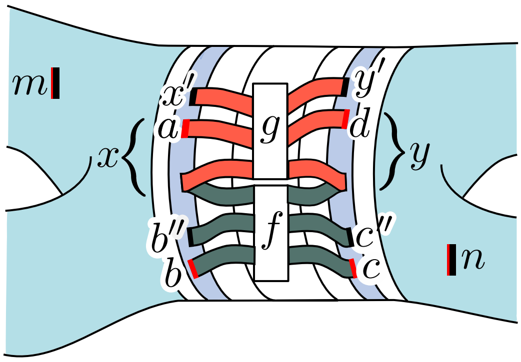
[†] We use s as this u_i may only be part of one of the ribbon diagrams in the decomposition of u.

$u_{i+1} := u_{[t-\epsilon,t+\epsilon]}$ be a crossing ribbon such that $v : b \sqcup c \to b' \sqcup c'$. There is an equivalence:

$$\left(v \sqcup \mathrm{Id}_{F(m \lhd x, y \rhd n)}\right) \circ \left(g \sqcup h \sqcup \mathrm{Id}_{F(b,c)}\right) \sim \left(g \sqcup h \sqcup \mathrm{Id}_{F(b',c')}\right) \circ \left(v \sqcup \mathrm{Id}_{F(m \lhd a, b \rhd n)}\right)$$

which commutes these ribbon diagrams up to some modification of the identity components.

Merging crossings. Let u_i and u_{i+1} both be crossing ribbon diagrams[†], so $u_i = f \sqcup \mathrm{Id}_{F(m \lhd a, d \rhd n)}$ and $u_{i+1} = g \sqcup \mathrm{Id}_{n \lhd b'', c'' \rhd m}$ for $f : b \sqcup c \to b' \sqcup c'$ and $g : x \sqcup y \to x' \sqcup y'$ for $x = a \sqcup (b' - b'')^*$ and $y = d \sqcup (c' - c'')$, see the figure below. Then the composition $u_{i+1} \circ u_i$ is equivalent to the single crossing $u' := v \sqcup \mathrm{Id}_{F(m,n)}$ where $v = (g \sqcup \mathrm{Id}_{b'' \sqcup c''}) \circ (f \sqcup \mathrm{Id}_{a \sqcup d})$.



We shall now check that the map $u \mapsto w$ is well defined by showing it is invariant under the equivalences listed above.

Equivalence of a non-crossing morphism This is straightforward: $u_i := f \sqcup g \mapsto \left(E_M^{-1}(f), E_N^{-1}(g)\right)$ and $u_i' := f' \sqcup g' \mapsto \left(E_M^{-1}(f'), E_N^{-1}(g')\right)$, but $f \sim f'$ and $g \sim g'$ implies $E_M^{-1}(f) \sim E_M^{-1}(f')$ and $E_N^{-1}(g) \sim E_N^{-1}(g')$, so these ribbon tangles map to the same morphism.

[†]To simplify the proof slightly, we assume that there are no points in the left crossing region which are not moved by the crossing.

$^*(b' - b'')$ denotes set difference

Equivalence of middle region

$$
\begin{tikzcd}
(m \lhd (a \sqcup b), (c \sqcup d) \rhd n) \arrow[r, "(\mathrm{Id}_m \lhd (\mathrm{Id}_a \sqcup p), (q \sqcup \mathrm{Id}_d) \rhd \mathrm{Id}_n)"] \arrow[d, "(\rho_{m,a,b}, \mathrm{Id})"'] \arrow[dr, phantom, "\text{naturality of } \rho"] & (m \lhd (a \sqcup x), (y \sqcup d) \rhd n) \arrow[d, "(\rho_{m,a,x}, \mathrm{Id})"] \\
((m \lhd a) \lhd b, (c \sqcup d) \rhd n) \arrow[r, "(\mathrm{Id}_{m \lhd a} \lhd p), (q \sqcup \mathrm{Id}_d) \rhd \mathrm{Id}_n)"] \arrow[d, "\iota_{m \lhd a, b, (c \sqcup d) \rhd n}"'] \arrow[dr, phantom, "\text{naturality of } \iota"] & ((m \lhd a) \lhd x, (y \sqcup d) \rhd n) \arrow[d, "\iota_{m \lhd a, x, (y \sqcup d) \rhd n}"] \\
(m \lhd a, b \rhd ((c \sqcup d) \rhd n)) \arrow[r, "(\mathrm{Id}, p \rhd ((q \sqcup \mathrm{Id}_d) \rhd \mathrm{Id}_n))"] \arrow[d, "(\mathrm{Id}, \hat{u}_1)"'] \arrow[dr, phantom, "\text{definition of maps}"] & (m \lhd a, x \rhd ((y \sqcup d) \rhd n)) \arrow[d, "(\mathrm{Id}, \hat{u}_2)"] \\
(m \lhd a, b' \rhd ((c' \sqcup d) \rhd n)) \arrow[r, "(\mathrm{Id}_{m \lhd a, r \rhd ((s \sqcup \mathrm{Id}_d) \rhd \mathrm{Id}_n)})"] \arrow[d, "\iota^{-1}_{m \lhd a, b', (c' \sqcup d) \rhd n}"'] \arrow[dr, phantom, "\text{naturality of } \iota"] & (m \lhd a, x' \rhd ((y' \sqcup d) \rhd n)) \arrow[d, "\iota^{-1}_{m \lhd a, x', (y' \sqcup d) \rhd n}"] \\
((m \lhd a) \lhd b', (c' \sqcup d) \rhd n) \arrow[r, "(\mathrm{Id}_{m \lhd a} \lhd r, (s \sqcup \mathrm{Id}_d) \rhd \mathrm{Id}_n)"] \arrow[d, "(\rho^{-1}_{m,a,b'}, \mathrm{Id})"'] \arrow[dr, phantom, "\text{naturality of } \rho"] & ((m \lhd a) \lhd x', (y' \sqcup d) \rhd n) \arrow[d, "(\rho^{-1}_{m,a,x'}, \mathrm{Id})"] \\
(m \lhd (a \sqcup b'), (c' \sqcup d) \rhd n) \arrow[r, "(\mathrm{Id}_m \lhd (\mathrm{Id}_a \sqcup r), (s \sqcup \mathrm{Id}_d) \rhd \mathrm{Id}_n)"] & (m \lhd (a \sqcup x'), (y' \sqcup d) \rhd n)
\end{tikzcd}
$$

Crossings commute with disjoint morphisms

$$\begin{tikzcd}
(m \lhd (a \sqcup b), (c \sqcup d) \rhd n)) \arrow[rr, "(g \sqcup \mathrm{Id}_{\emptyset \lhd b}, (\mathrm{Id}_{c \rhd \emptyset} \sqcup h))"] \arrow[d, "(\rho_{m,a,b}, \mathrm{Id})"'] \arrow[drr, phantom, "\text{naturality of } \rho"] & & (m \lhd (x \sqcup b), (c \sqcup y) \rhd n)) \arrow[d, "(\rho_{m,x,b}, \mathrm{Id})"] \\
((m \lhd a) \lhd b, (c \sqcup d) \rhd n)) \arrow[rr, "(g \lhd \mathrm{Id}_b, \mathrm{Id}_{c \rhd \emptyset} \sqcup h)"] \arrow[d, "\iota_{m \lhd a, b, (c \sqcup d) \rhd n}"'] \arrow[drr, phantom, "\text{naturality of } \iota"] & & ((m \lhd x) \lhd b, (c \sqcup y) \rhd n)) \arrow[d, "\iota_{m \lhd x, b, (c \sqcup y) \rhd n}"] \\
(m \lhd a, b \rhd ((c \sqcup d) \rhd n)) \arrow[rr, "(g, \mathrm{Id}_b \rhd (\mathrm{Id}_{c \rhd \emptyset} \sqcup h))"] \arrow[d, "(\mathrm{Id}, \overline{v} \sqcup \mathrm{Id}_{\emptyset \rhd (d \rhd n)})"'] \arrow[drr, phantom, "\text{identity morphisms are central}"] & & (m \lhd x, b \rhd ((c \sqcup y) \rhd n)) \arrow[d, "(\mathrm{Id}, \overline{v} \sqcup \mathrm{Id}_{\emptyset \rhd (y \rhd n)})"] \\
(m \lhd a, b' \rhd ((c' \sqcup d) \rhd n)) \arrow[rr, "(g, \mathrm{Id}_{b'} \rhd (\mathrm{Id}_{c' \rhd \emptyset} \sqcup h))"] \arrow[d, "\iota^{-1}_{m \lhd a, b', (c' \sqcup d) \rhd n}"'] \arrow[drr, phantom, "\text{naturality of } \iota"] & & (m \lhd x, b' \rhd ((c' \sqcup y) \rhd n)) \arrow[d, "\iota^{-1}_{m \lhd x, b', (c' \sqcup y) \rhd n}"] \\
((m \lhd a) \lhd b', (c' \sqcup d) \rhd n) \arrow[rr, "(g \lhd \mathrm{Id}_{b'}, \mathrm{Id}_{c' \rhd \emptyset} \sqcup h)"] \arrow[d, "(\rho^{-1}_{m,a,b'}, \mathrm{Id})"'] \arrow[drr, phantom, "\text{naturality of } \rho"] & & ((m \lhd x) \lhd b', (c' \sqcup y) \rhd n) \arrow[d, "(\rho^{-1}_{m,x,b'}, \mathrm{Id})"] \\
(m \lhd (a \sqcup b'), (c' \sqcup d) \rhd n) \arrow[rr, "(g \sqcup \mathrm{Id}_{\emptyset \lhd b'}, (\mathrm{Id}_{c' \rhd \emptyset} \sqcup h))"] & & (m \lhd (x \sqcup b'), (c' \sqcup y) \rhd n)
\end{tikzcd}$$

Merging Crossings

$$
\begin{tikzcd}
(m \lhd (a \sqcup b), (c \sqcup d \sqcup e) \rhd n) \arrow[rr, "\iota_{m,a \sqcup b,(c \sqcup d \sqcup e \sqcup f) \rhd n}"] \arrow[d, "{(\rho_{m,a,b}, \mathrm{Id})}"] & & (m, (a \sqcup b) \rhd ((c \sqcup d \sqcup e) \rhd n)) \arrow[d, "{(\mathrm{Id}, \rho_{a,b,(c \sqcup d \sqcup e) \rhd n})}"] \\
((m \lhd a) \lhd b, (c \sqcup d \sqcup e) \rhd n) \arrow[dr, "\iota_{m \lhd a,b,(c \sqcup d \sqcup e) \rhd n}"'] & \text{pentagon} & (m, a \rhd (b \rhd ((c \sqcup d \sqcup e) \rhd n))) \arrow[dl, "\iota^{-1}_{m,a,b \rhd ((c \sqcup d \sqcup e) \rhd n)}"] \arrow[ddd, "{(\mathrm{Id}_m, \mathrm{Id}_a \rhd \hat{f})}"] \\
& (m \lhd a, b \rhd ((c \sqcup d \sqcup e) \rhd n)) \arrow[d, "{(\mathrm{Id}_m \lhd \mathrm{Id}_a, \hat{f})}"] \arrow[r, phantom, "\iota \text{ naturality}"] & \\
& (m \lhd a, b' \rhd ((c' \sqcup d \sqcup e) \rhd n)) \arrow[dl, "\iota^{-1}_{m \lhd a,b',(c' \sqcup d \sqcup e) \rhd n}"'] \arrow[dr, "\iota_{m,a,b' \rhd ((c' \sqcup d \sqcup e) \rhd n)}"] & \\
((m \lhd a) \lhd b', (c' \sqcup d \sqcup e) \rhd n) \arrow[d, "{(\rho^{-1}_{m,a,b'}, \mathrm{Id})}"] & \text{pentagon} & (m, a \rhd (b' \rhd ((c' \sqcup d \sqcup e) \rhd n))) \arrow[d, "{(\mathrm{Id}, \rho^{-1}_{a,b',(c' \sqcup d \sqcup e) \rhd n})}"] \\
(m \lhd (a \sqcup b')), (c' \sqcup d \sqcup e) \rhd n) \arrow[rr, "\iota_{m,a \sqcup b',(c' \sqcup d \sqcup e) \rhd n}"] \arrow[d, equal] \arrow[dr, phantom, "="] & & (m, (a \sqcup b') \rhd (c' \sqcup d \sqcup e) \rhd n) \arrow[d, equal] \\
(m \lhd (b'' \sqcup x)), (y \sqcup c'' \sqcup e) \rhd n) \arrow[rr, "\iota_{m,b'' \sqcup x,(y \sqcup c'' \sqcup e) \rhd n}"] \arrow[d, "{(\rho_{m,b'',x}, \mathrm{Id})}"] & & (m, (b'' \sqcup x) \rhd (y \sqcup c'' \sqcup e) \rhd n) \arrow[d, "{(\mathrm{Id}, \rho_{b'',x,(y \sqcup c'' \sqcup e) \rhd n})}"] \\
((m \lhd b'') \lhd x, (y \sqcup c'' \sqcup e) \rhd n) \arrow[dr, "\iota_{m \lhd b'',x,(y \sqcup c'' \sqcup e) \rhd n}"'] & \text{pentagon} & (m, b'' \rhd (x \rhd (y \sqcup c'' \sqcup e) \rhd n)) \arrow[dl, "\iota^{-1}_{m,b'',(y \sqcup c'' \sqcup e) \rhd n}"] \arrow[ddd, "{(\mathrm{Id}_m, \mathrm{Id}_{b''} \rhd \hat{g})}"] \\
& (m \lhd b'', x \rhd ((y \sqcup c'' \sqcup e) \rhd n)) \arrow[d, "{(\mathrm{Id}_m \lhd \mathrm{Id}_{b''}, \hat{g})}"] \arrow[r, phantom, "\iota \text{ naturality}"] & \\
& (m \lhd b'', x' \rhd ((y' \sqcup c'' \sqcup e) \rhd n)) \arrow[dl, "\iota^{-1}_{m \lhd b'',x',(y' \sqcup c'' \sqcup e) \rhd n}"'] \arrow[dr, "\iota_{m,b'',x' \rhd ((y' \sqcup c'' \sqcup e) \rhd n)}"] & \\
((m \lhd b'') \lhd x', (y' \sqcup c'' \sqcup e) \rhd n) \arrow[d, "{(\rho^{-1}_{m,b'',x'}, \mathrm{Id})}"] & \text{pentagon} & (m, b'' \rhd (x' \rhd ((y' \sqcup c'' \sqcup e) \rhd n))) \arrow[d, "{(\mathrm{Id}, \rho^{-1}_{b'',x',(y' \sqcup c'' \sqcup e) \rhd n})}"] \\
(m \lhd (b'' \sqcup x'), (y' \sqcup c'' \sqcup e) \rhd n) & & (m, (b'' \sqcup x') \rhd ((y' \sqcup c'' \sqcup e) \rhd n)) \arrow[ll, "\iota^{-1}_{m,b'' \sqcup x',(y' \sqcup c'' \sqcup e) \rhd n}"]
\end{tikzcd}
$$

The composition of the morphism on the right of the diagram is $u_{i+1} \circ u_i$, so

$$
\begin{aligned}
u_{i+1} \circ u_i = {} & \iota^{-1}_{m,b'' \sqcup x',(y' \sqcup c'' \sqcup e) \rhd n} \\
& \circ (\mathrm{Id}, \rho^{-1}_{b'',x',(y' \sqcup c'' \sqcup e) \rhd n}) \circ (\mathrm{Id}_m, \mathrm{Id}_{b''} \rhd \hat{g}) \circ (\mathrm{Id}, \rho_{b'',x,(y \sqcup c'' \sqcup e) \rhd n}) \\
& \circ (\mathrm{Id}, \rho^{-1}_{a,b',(c' \sqcup d \sqcup e) \rhd n}) \circ (\mathrm{Id}_m \lhd \mathrm{Id}_a, \hat{f}) \circ (\mathrm{Id}, \rho_{a,b,(c \sqcup d \sqcup e) \rhd n})
\end{aligned}
$$

$$
\begin{aligned}
&\circ\, \iota_{m,a\sqcup b,(c\sqcup d\sqcup e\sqcup f)\rhd n}\\
&= \iota^{-1}_{m,b''\sqcup x',(y'\sqcup c''\sqcup e)\rhd n}\\
&\quad \circ (\mathrm{Id}, \mathrm{Id}_{b''\rhd\emptyset} \sqcup \hat{g}) \circ (\mathrm{Id}, \rho^{-1}_{b'',x,(y\sqcup c''\sqcup e)\rhd n}) \circ (\mathrm{Id}, \rho_{b'',x,(y\sqcup c''\sqcup e)\rhd n})\\
&\quad \circ (\mathrm{Id}, \mathrm{Id}_{a\rhd\emptyset} \sqcup \hat{f}) \circ (\mathrm{Id}, \rho^{-1}_{a,b,(c\sqcup d\sqcup e)\rhd n}) \circ (\mathrm{Id}, \rho_{a,b,(c\sqcup d\sqcup e)\rhd n})\\
&\quad \circ\, \iota_{m,a\sqcup b,(c\sqcup d\sqcup e\sqcup f)\rhd n}\\
&\quad \text{by naturality of } \rho\\
&= \iota^{-1}_{m,b''\sqcup x',(y'\sqcup c''\sqcup e)\rhd n} \circ (\mathrm{Id}, \mathrm{Id}_{b''\rhd\emptyset} \sqcup \hat{g}) \circ (\mathrm{Id}, \mathrm{Id}_{a\rhd\emptyset} \sqcup \hat{f}) \circ \iota_{m,a\sqcup b,(c\sqcup d\sqcup e\sqcup f)\rhd n}\\
&= u'
\end{aligned}
$$

as required. □

4.3 Relation to Factorisation Homology

4.3.1 Skein Categories and k–linear Factorisation Homologies

Fix a k–linear strict ribbon category $\mathscr{V}$. We shall now use the results proven so far in this chapter to conclude that the skein category $\mathbf{Sk}_{\mathscr{V}}(\Sigma)$ is the k–linear factorisation homology $\int_\Sigma \mathscr{V}$.

I. As $\mathscr{V}$ is a braided monoidal category it defines an E_2–algebra.

II. We saw in Remark 4.2.6 that an embedding of surfaces $\Sigma \hookrightarrow \Pi$ induces a functor $\mathbf{Sk}(\Sigma) \to \mathbf{Sk}(\Pi)$ between their skein categories, and in Remark 4.2.12 that isotopies of embeddings define natural transformations. This implies that

$$\mathbf{Sk}_{\mathscr{V}}(\_) : \mathbf{Mfld}^2_{\mathrm{fr}} \to \mathbf{Cat}_k$$

is a 2–functor.

III. From Corollary 4.2.10 we have an equivalence of categories $\mathbf{Sk}_{\mathscr{V}}(\mathbb{D}^2) \simeq \mathscr{V}$.

IV. From Remark 4.2.7 we have for any 1–manifold C that $\mathbf{Sk}(C \times [0,1])$ has a canonical monoidal structure induced from the inclusions of intervals.

V. From Theorem 4.2.16 we have given suitable thickened embeddings an equivalence of categories

$$\mathbf{Sk}_{\mathscr{V}}(M \sqcup_A N) \simeq \mathbf{Sk}_{\mathscr{V}}(M) \times_{\mathbf{Sk}_{\mathscr{V}}(A)} \mathbf{Sk}_{\mathscr{V}}(N).$$

As a factorisation homology is fully characterised by the above (Theorem 2.3.13), we conclude:

Theorem 4.3.1. *Let $\mathscr{V}$ be k–linear strict ribbon category $\mathscr{V}$. The functor*

$$\mathbf{Sk}_{\mathscr{V}}(\_) : \mathbf{Mfld}^2_{\mathrm{fr}} \to \mathbf{Cat}_k$$

is the k–linear factorisation homology

$$\int_{\_} \mathscr{V} : \mathbf{Mfld}^2_{\mathrm{fr}} \to \mathbf{Cat}_k$$

of surfaces with coefficients in $\mathscr{V}$.

4.3.2 Skein Categories and Presentable Factorisation Homologies

Finally, we shall use the relation between $\mathbf{Pr}$ and $\mathbf{Cat}_k$ to show that one can freely cocomplete a skein category to recover a presentable factorisation homology. Before we do this we must introduce one more category $\mathbf{Ico}$, the $(2,1)$–category of idempotent complete categories.

Idempotent Complete Categories

Definition 4.3.2. A morphism $e : x \to x$ is an *idempotent* if $e \circ e = e$.

Definition 4.3.3. A *retract* of the object $x \in \mathscr{C}$ is an object $y \in \mathscr{C}$ and morphisms

$$y \underset{r}{\overset{i}{\rightleftarrows}} x$$

such that $r \circ i = \mathrm{Id}_y$. Note that $r \circ i$ is an idempotent.

Definition 4.3.4. An idempotent $e : x \to x$ *splits* if there is a retract $y \underset{r}{\overset{i}{\rightleftarrows}} x$ such that $r \circ i = e$. A category $\mathscr{C}$ is *idempotent complete* or *Cauchy complete* if all idempotents in $\mathscr{C}$ split.

As any functor preserves idempotents and their splittings, we make the following definition:

Definition 4.3.5. The *category of idempotent complete k–linear categories* $\mathbf{Ico}$ is the $(2,1)$–category whose

1. objects are small idempotent complete categories;
2. 1–morphisms are k–linear functors;
3. 2–morphisms are k–linear natural isomorphisms.

Idempotent complete categories may also be characterised in terms of absolute colimits.

Definition 4.3.6. A weighted colimit $\mathrm{Colim}_G(F)$ is an *absolute colimit* if it is preserved by all functors.

The idempotent $e : x \to x$ splits if and only if the equaliser $\mathrm{Ker}(e, \mathrm{Id}_x)$ and the coequaliser $\mathrm{Coker}(\mathrm{Id}_x, e)$ exist. In which case $i = \mathrm{Ker}(e, \mathrm{Id}_x)$ and $r = \mathrm{Coker}(\mathrm{Id}_x, e)$ and they are absolute colimits. Hence,

Proposition 4.3.7 [Bor94a]**.** *Let $\mathscr{C}$ be a small category. The following conditions are equivalent:*

1. *$\mathscr{C}$ is idempotent complete;*
2. *$\mathscr{C}$ has all absolute colimits.*

Definition 4.3.8 [BD86]**.** Let $\mathscr{C}$ be a small $\mathscr{V}$-enriched category. The *idempotent completion* or *Cauchy completion* of $\mathscr{C}$ is the full subcategory of the $\mathscr{V}$–enriched presheaf category $\mathbf{PSh}^{\mathscr{V}}(\mathscr{C})$ consisting of absolute colimits of representable functors. It is denoted $\mathrm{Ico}(\mathscr{C})$.

Remark 4.3.9. If $\mathscr{C}$ is small then so is $\mathrm{Ico}(\mathscr{C})$, and $\mathrm{Ico}(\mathscr{C}) \simeq \mathscr{C}$ if and only if $\mathscr{C}$ is idempotent complete.

Relations between $\mathbf{Cat}_k$, Ico and Pr

We now recall a few results which relate categories in Cat_k, **Ico** and **Pr**.

Proposition 4.3.10 [Bor94b]**.** *Idempotent completion defines a functor of k–linear monoidal categories*

$$\mathrm{Ico} : \mathbf{Cat}_k \to \mathbf{Ico}$$

Definition 4.3.11. Let $\mathscr{C}$ be a small category. The *free cocompletion* $\mathrm{Free}(\mathscr{C})$ is given by the Yoneda embedding $Y : \mathscr{C} \to \mathbf{PSh}(\mathscr{C})^\dagger$.

Proposition 4.3.12 [AR94]. *The free cocompletion* $\mathrm{Free}(\mathscr{C})$ *of a small k–linear category is locally finitely presentable.*

Proposition 4.3.13 [KS06]**.** *The free cocompletion of categories defines a bicolimit preserving functor of k–linear monoidal categories*

$$\mathrm{Free} : \mathbf{Cat}_k \to \mathbf{Pr}.$$

Definition 4.3.14. An object $c \in \mathscr{C}$ of a category $\mathscr{C}$ is *compact-projective* if the corepresentable functor $\mathscr{C}(c, \_) : \mathscr{C} \to \mathscr{V}$ preserves all small colimits.

Proposition 4.3.15 [BD86]**.** *There is a functor of k–linear monoidal categories*

$$\mathrm{Comp} : \mathbf{Pr} \to \mathbf{Ico}$$

which sends $\mathscr{C}$ to its full subcategory $\mathrm{Comp}(\mathscr{C})$ *of compact-projective objects.*

Proposition 4.3.16 [BD86]**.** *The functors* Free *and* Comp *satisfy the relations that for any $\mathscr{C} \in \mathbf{Cat}_k$:*

$$\mathrm{Comp}(\mathrm{Free}(\mathscr{C})) \simeq \mathrm{Ico}(\mathscr{C})$$

and for any $\mathscr{D} \in \mathbf{Pr}$:

$$\mathrm{Free}(\mathrm{Comp}(\mathscr{D})) \simeq \mathscr{D}.$$

Conclusion

Using the results just stated and that $\mathbf{Sk}_{\mathscr{V}}(\Sigma) = \int_{\Sigma}^{\mathbf{Cat}_k} \mathscr{V}$ we conclude:

Theorem 4.3.17. *There are equivalences of categories*

$$\mathrm{Free}(\mathbf{Sk}(\mathscr{V})) \simeq \int_S^{\mathbf{Pr}} \mathrm{Free}(\mathscr{V}) \textit{ and } \mathrm{Comp}\left(\int_S^{\mathbf{Pr}} \mathrm{Free}(\mathscr{V})\right) \simeq \mathrm{Ico}(\mathbf{Sk}((\mathscr{V})),$$

*so in particular**

$$\mathrm{Free}(\mathbf{Sk}(\mathbf{Rep}_q^{\mathrm{fd}}(G))) \simeq \int_S^{\mathbf{Pr}} \mathbf{Rep}_q(G) \textit{ and } \mathrm{Comp}\left(\int_S^{\mathbf{Pr}} \mathbf{Rep}_q(G)\right) \simeq \mathrm{Ico}(\mathbf{Sk}(\mathbf{Rep}_q^{\mathrm{fd}}(G))).$$

† Technically the free cocompletion is defined in terms of a universal property and then shown in this case to be given by the Yoneda embedding, see [AR94] for details.

* Note that as Ico commutes with finite bicolimits, so if we define $\overline{\mathbf{Sk}}(\mathscr{V}) := \mathrm{Ico}(\mathbf{Sk}(\mathscr{V}))$ then we still have excision: $\overline{\mathbf{Sk}}(M) \times_{\overline{\mathbf{Sk}}(A)} \overline{\mathbf{Sk}}(N) \simeq \overline{\mathbf{Sk}}(M \sqcup_A N)$.

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